Reconstruction of Non-Stationary Signals by the Generalized Prony Method

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We derive a method for the reconstruction of non-stationary signals with structured phase functions using only a small number of signal measurements. Our approach employs generalized shift operators as well as the generalized Prony method. Our goal is to reconstruct a variety of sparse signal models using a small number of signal measurements.

1 Introduction

We consider the problem of recovering structured functions of the form

$$f(x) = \sum_{j=1}^{M} c_j H(x) \mathrm{e}^{\alpha_j G(x)} \tag{1}$$

where $G\colon [a,b] \to \mathbb{R}$ is a known continuous and strictly monotone phase function and $H\colon \mathbb{R} \to \mathbb{C}$ is a known continuous function that has no zeros in $[a,b]$. Signals of the form (1) are called non-stationary if $H(x)$ is not a constant function and/or the phase function $G(x)$ is not of the form $mx+d$ with $m, d \in \mathbb{R}$. For the special case $H(x) \equiv 1$ and $G(x) = x$ this reconstruction problem can be solved with the Prony method [3] thereby using only $2M$ functional values.

2 Generalized Shift Operators and the Prony method

The generalized Prony method in [1, 2, 4] enables us to reconstruct sparse expansions of eigenfunctions of a linear operator. Therefore, we try to find a linear shift operator possessing eigenfunctions of the form $H(x)\,\mathrm{e}^{\alpha_j G(x)}$.

For $h \in \mathbb{R} \setminus \{0\}$, we consider the following generalized shift operator

$$S_{H,G,h} f(x) := \frac{H(x)}{H\left(G^{-1}\left(G(x)+h\right)\right)} f\left(G^{-1}\left(G(x)+h\right)\right). \tag{2}$$

Theorem 2.1 *Let $S_{H,G,h}$ be of the form (2) with H and G as in (1). Then $S_{H,G,h}$ possesses eigenfunctions of the form $H(x)\,\mathrm{e}^{\alpha G(x)}$ corresponding to the eigenvalue $\mathrm{e}^{\alpha h}$ for $\alpha \in \mathbb{R}$.*

P r o o f. Employing the definition of $S_{H,G,h}$ yields

$$\begin{aligned} S_{H,G,h}\left(H(\cdot)\,\mathrm{e}^{\alpha G(\cdot)}\right)(x) &= \frac{H(x)}{H\left(G^{-1}\left(G(x)+h\right)\right)} H\left(G^{-1}\left(G(x)+h\right)\right)\, \mathrm{e}^{\alpha G\left(G^{-1}(G(x)+h)\right)} \\ &= H(x)\, e^{\alpha(G(x)+h)} = \mathrm{e}^{\alpha h}\, H(x)\, \mathrm{e}^{\alpha G(x)}, \end{aligned}$$

i.e., $H(x)\,\mathrm{e}^{\alpha G(x)}$ is an eigenfunction of $S_{H,G,h}$ corresponding to the eigenvalue $\mathrm{e}^{\alpha h}$. □

Theorem 2.2 *Let f be of the form (1). Then f can be uniquely reconstructed from the function values $f\left(G^{-1}\left(G(x_0)+kh\right)\right)$ for $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ and $h \in \mathbb{R}\setminus\{0\}$ are chosen such that $G(x_0)+kh$ is in the domain of G^{-1}.*

P r o o f. We define the Prony polynomial $P(z) := \prod_{j=1}^{M}(z - e^{\alpha_j h}) = \sum_{k=0}^{M} p_k z^k$. In a first step, we want to recover this polynomial from the given function values. We use Theorem 2.1 and observe for $m = 0, \ldots, M-1$,

$$\begin{aligned} \sum_{k=0}^{M} p_k S_{H,G,h}^{(k+m)} f(x_0) &= \sum_{k=0}^{M} p_k S_{H,G,h}^{(k+m)} \left(\sum_{j=1}^{M} c_j H(x_0) \mathrm{e}^{\alpha_j G(x_0)}\right) = \sum_{j=1}^{M} c_j \sum_{k=0}^{M} p_k S_{H,G,h}^{(k+m)} \left(H(x_0)\mathrm{e}^{\alpha_j G(x_0)}\right) \\ &= \sum_{j=1}^{M} c_j \sum_{k=0}^{M} p_k\, \mathrm{e}^{\alpha_j h(k+m)} H(x_0) \mathrm{e}^{\alpha_j G(x_0)} = \sum_{j=1}^{M} c_j H(x_0) \mathrm{e}^{\alpha_j G(x_0)} \mathrm{e}^{\alpha_j hm} P(\mathrm{e}^{\alpha_j h}) = 0. \end{aligned}$$

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Exploiting that $p_M = 1$, and that $S^k_{H,G,h} f = S_{H,G,kh} f$, we derive the linear system $\mathbf{Hp} = -\mathbf{f}_M$ with the vector $\mathbf{p} = (p_0, \ldots, p_{M-1})^T$ of coefficients of the Prony polynomial and

$$\mathbf{H} := \left(d_{k+m}\, f(G^{-1}(G(x_0) + (k+m)h))\right)_{k,m=0}^{M-1}, \qquad \mathbf{f}_M = \left(d_{k+M}\, f(G^{-1}(G(x_0) + (k+M)h))\right)_{k=0}^{M-1},$$

where $d_\ell := \frac{H(x_0)}{H(G^{-1}(G(x_0)+\ell h))}$, $\ell = 0, \ldots, 2M-1$ can be precomputed. The Hankel matrix $\mathbf{H}$ admits the factorization

$$\mathbf{H} = H(x_0)\, \mathbf{V}_\lambda\, \mathrm{diag}(c_1 \mathrm{e}^{\alpha_1 G(x_0)}, \ldots, c_M \mathrm{e}^{\alpha_M G(x_0)})\, \mathbf{V}_\lambda^T$$

with the Vandermonde matrix $\mathbf{V}_\lambda = \left(\mathrm{e}^{\alpha_j hk}\right)_{k=0,j=1}^{M-1,M}$. Since $H(x_0) \neq 0$ and $\mathbf{V}_\lambda$ has full rank, we conclude that $\mathbf{H}$ is invertible. Having found the coefficients p_k of the Prony polynomial, we can compute its roots $\mathrm{e}^{\alpha_j h}$, $j = 1, \ldots, M$, and then determine the parameters c_j by solving the linear system

$$\frac{1}{H\left(G^{-1}(G(x_0) + hk)\right)} f\left(G^{-1}(G(x_0) + hk)\right) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j hk}\, \mathrm{e}^{\alpha_j G(x_0)}, \qquad k = 0, \ldots, 2M-1.$$

□

The idea can be extended even further using symmetric generalized shift operators.

Corollary 2.3 *Let* $f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x^p + \beta_j)$ *with given odd integer* $p > 0$*, and unknown coefficients* $c_j \in \mathbb{R} \setminus \{0\}$, $\beta_j \in [0, \pi] \setminus \{\frac{\pi}{2}\}$*, and pairwise different* $\alpha_j \in [0, K)$ *for some* $K > 0$ *for all* $j = 1, \ldots, M$*. Then* α_j, β_j, c_j, $j = 1, \ldots, M$, *can be reconstructed from* $f(\pm\sqrt[p]{kh})$, $k = 0, \ldots, 2M-1$, *where* $0 < h \leq \frac{\pi}{2K}$.

P r o o f. For a detailed proof for the recovery of the α_j,$j = 1, \ldots, M$ see [2]. For the recovery of the c_j and β_j we use that $\cos(x+y) - \cos(x-y) = -2\sin(x)\sin(y)$ and that $\widetilde{V} = (\sin(\alpha_j lh))_{l=0,j=1}^{M-1,M}$ is invertible for α_j and h for $j = 1, \ldots, M$ as above. □

3 Numerical Example

We illustrate the recovery method in Corollary 2.3 with a numerical example. Let $f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x^p + \beta_j)$ with $M = 2$, $p = 3$, $\alpha_1 = 2.5305$, $\alpha_2 = 1.8118$, $c_1 = 0.9146$, $c_2 = 1.1997$ and $\beta_1 = 0.5378$, $\beta_2 = 2.0592$. We use the 7 sample values $f(\pm\sqrt[3]{k})$ for $k = 0, \ldots, 3$.
The reconstruction errors are

$$\max_j |c_j - \tilde{c}_j| = 1.998 \cdot 10^{-15}, \qquad \max_j |\alpha - \tilde{\alpha}_j| = 1.33 \cdot 10^{-15}, \qquad \max_j |\beta_j - \tilde{\beta}_j| = 1.44 \cdot 10^{-15}.$$

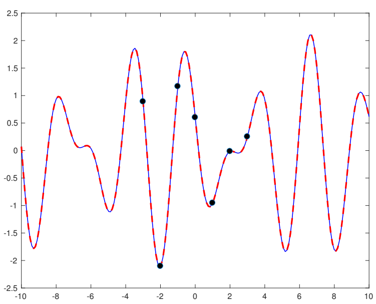

Fig. 1: The blue line represents the original signal. The reconstructed signal is plotted in red.The black dots indicate the used signal values of f.

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References

[1] T. Peter and G. Plonka, Inverse Problems **29**, (2013).
[2] G. Plonka, K. Stampfer, and I. Keller, Anal. Appl. **17**(2), 179–210, (2019).
[3] G. Plonka and M. Tasche, GAMM-Mitt. **37**(2), 239–258, (2014).
[4] K. Stampfer and G. Plonka, arXiv:1901.08778, (2019)