Crossed Products in the Correspondence Category

Diplomarbeit

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Abstract

A 2-category generalizes the concept of a category by adding 2-morphisms between arrows. The easiest example would be the category of small categories, with functors between categories as the usual morphisms, and natural equivalences between functors as 2-morphisms. One can weaken the notion of a 2-category to a weak 2-category. That is, the unit and associativity laws for concatenation of arrows are only required to hold up to 2-morphisms.
An example for a weak 2-category is the correspondence 2-category $\mathfrak{Corr}(2)$, which consist of C^*-algebras as objects, C^*-correspondences as morphisms and C^*-correspondence isomorphisms as 2-morphisms. Since invertible C^*-correspondences are imprimitivity bimodules, isomorphisms in $\mathfrak{Corr}(2)$ becomes Morita equivalences.
Given an arbitrary weak 2-category, one can define weak actions of groups, 2-groups, groupoids or 2-groupoids on objects in the given category as 2-functors. That is, given one of the former mentioned structures, every property of the action is already encoded in the definition of the 2-functor. An interesting fact about the correspondence 2-category is, that a group action α of a locally compact group G on a C^*-algebra A in $\mathfrak{Corr}(2)$ is equivalent to a saturated Fell bundle $\mathfrak{A}$, such that A is isomorphic to the unit fiber A_1.
Using the equivalence between crossed modules and 2-groups and the fact, that crossed modules correspond in some way to quotient groups, one can generalize this equivalence to 2-groups by defining a Fell bundle over a 2-group in a way, that it is isomorphic two a pull-back bundle. Again, finding the right notion of continuity for 2-group actions in $\mathfrak{Corr}(2)$ yields that this equivalence preserves continuity.
Moreover, the results on discrete group actions can be easily extended two r-discrete groupoids. And since there exist quotients and crossed modules over groupoids, one can generalize the construction for discrete 2-groups to find a notion of Fell bundles over- and weak actions of r-discrete 2-groupoids in $\mathfrak{Corr}(2)$. These can be showed to be still equivalent.

Contents

Chapter 1

The $\mathfrak{Corr}(2)$ 2-Category

In this chapter, we are going to construct the 2-category $\mathfrak{Corr}(2)$. We are starting with a quick introduction of the basic theory of *imprimitivity bimodules*, mostly based on the work of Raeburn, Williams and Echterhoff ([8], [20]). Furthermore, we will define C^**-correspondences*, and establish the connection between the latter and imprimitivity bimodules. Finally we will introduce the notion of a (*weak*) *2-category*, discuss how to define actions of groups and groupoids as 2-functors between categories and use the previous results to define the *correspondence 2-category* $\mathfrak{Corr}(2)$.

1.1 Imprimitivity modules

Definition 1.1. Let A be a C^*-algebra, a (*right*) *Hilbert A-module* $\mathcal{E}_A$ is a right A-module with a scalar multiplication

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \qquad \forall x \in \mathcal{E}_A, a \in A, \lambda \in \mathbb{C}$$

and a map

$$\langle \cdot, \cdot \rangle : \mathcal{E}_A \times \mathcal{E}_A \to A$$

satisfying the following conditions

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$

(ii) $\langle x, ya \rangle = \langle x, y \rangle a$

(iii) $\langle y, x \rangle = \langle x, y \rangle^*$

(iv) $\langle x, x \rangle \geq 0 \quad \text{if } \langle x, x \rangle = 0 \Rightarrow x = 0$

for all $x, y, z \in \mathcal{E}_A$, $\alpha, \beta \in \mathbb{C}$ and $a \in A$, which is complete with respect to the scalar valued norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.
A Hilbert module is called *full*, if $\langle \mathcal{E}_A, \mathcal{E}_A \rangle$ is dense in A.

Example 1.2. Let A be a C^*-algebra. Then we can define an Hilbert module $\mathcal{A}_A$, where the right action is implemented by right multiplication, and the inner product is given by $\langle a, b \rangle_A := a^* b$.
From the existence of an approximate unit, it follows that $\mathcal{A}_A$ is even a full Hilbert A-module.

Definition 1.3. Let $\mathcal{E}_A$ be a Hilbert A-module, $x, y \in \mathcal{E}_A$. We define a *compact operator* on $\mathcal{E}_A$ by

$$\begin{aligned} \theta_{x,y} \colon \mathcal{E}_A &\to A \\ z &\mapsto x\langle y, z\rangle \end{aligned}$$

and denote the *set of compact operators* by

$$\mathcal{K}(\mathcal{E}_A) := \operatorname{span}\{\theta_{x,y} : x, y \in \mathcal{E}_A\}$$

Remark 1.4. Let A be a C^*-algebra, then $\mathcal{K}(\mathcal{A}_A) = A$. Again this can be shown easily by using the approximate unit of A.

Lemma 1.5. *[20] Let A be a C^*-algebra, $\mathcal{E}_A$ be an Hilbert A-module. Then $\mathcal{E}_A$ is a full left Hilbert $\mathcal{K}(\mathcal{E}_A)$-module with respect to the natural left action $T \cdot x = T(x)$ and the inner product ${}_{\mathcal{K}(\mathcal{E}_A)}\langle x, y\rangle := \theta_{x,y}$*

Proof. The proof is just some easy calculations and may be found in [20]. □

Definition 1.6. Let A, B be C^*-algebras. We call A and B *strongly Morita equivalent* if there is a full Hilbert A-module $\mathcal{E}_A$, such that $B \cong \mathcal{K}(\mathcal{E}_A)$.
We will denote the equivalence by $A \sim_{M.} B$.

Definition 1.7. Let A, B be C^*-algebras. An $A-B$*-imprimitivity module* ${}_A\mathcal{X}_B$ is an $A-B$-bimodule, which has to bilinear mappings

$$\begin{aligned} {}_A\langle \cdot, \cdot \rangle : {}_A\mathcal{X}_B \times {}_A\mathcal{X}_B \to A \\ \langle \cdot, \cdot \rangle_B : {}_A\mathcal{X}_B \times {}_A\mathcal{X}_B \to B \end{aligned}$$

such that ${}_A\mathcal{X}_B$ is a full left Hilbert A-module and a full right Hilbert B-module satisfying the following conditions

(i) ${}_A\langle x, y\rangle z = x\langle y, z\rangle_B$

(ii) $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$ and ${}_A\langle x \cdot b, y\rangle = {}_A\langle x, y \cdot b^*\rangle$

for all x, y, $z \in {}_A\mathcal{X}_B$

Remark 1.8. For the sake of convenience, we will omit the indices provided that it is clear about what kind of module we are talking.

Proposition 1.9. *[20] Every full Hilbert B-module is a $\mathcal{K}(\mathcal{E}_B)-B$-imprimitivity bimodule with ${}_{\mathcal{K}(\mathcal{E}_B)}\langle x, y\rangle := \theta_{x,y}$.
Conversely, if ${}_A\mathcal{X}_B$ is an $A-B$-imprimitivity bimodule, then there is an isomorphism $\phi\colon A \to \mathcal{K}({}_A\mathcal{X}_B)$ such that $\phi({}_A\langle x, y\rangle) = {}_{\mathcal{K}(\mathcal{E}_B)}\langle x, y\rangle$.*

Proof. [20] Suppose $\mathcal{E}_B$ is a full Hilbert B-module, Lemma 1.5 says that ${}_{\mathcal{K}(\mathcal{E}_B)}\mathcal{E}_B$ is a full left Hilbert $\mathcal{K}(\mathcal{E}_B)$. The first identity of condition (ii) of Definition 1.7 follows directly from the fact, that $\mathcal{K}(\mathcal{E}_B)$ acts by adjointable operators. For the second, let $b \in B$, $x, y, z \in \mathcal{E}_B$, and compute:

$${}_{\mathcal{K}(\mathcal{E}_B)}\langle x \cdot b, y\rangle(z) = (x \cdot b) \cdot \langle y, z\rangle_B = x \cdot \langle y \cdot b^*, z\rangle_B = {}_{\mathcal{K}(\mathcal{E}_B)}\langle x, y \cdot b^*\rangle(z)$$

The associativity condition is the definition of ${}_{\mathcal{K}(\mathcal{E}_B)}\langle x, y\rangle = \theta_{x,y}$, thus we have proved the first assumption.

Now suppose $\mathcal{X}$ is an $A-B$-imprimitivity bimodule. Since A acts by adjointable operators, the map $\phi\colon A \to \mathcal{L}(\mathcal{X})$, defined by $\phi(a)(x) := a \cdot x$ is a homomorphism of C^*-algebras, and hence has closed range. The associativity condition of Definition 1.7 implies that $\phi({}_A\langle x, y\rangle) = {}_{\mathcal{K}(\mathcal{X})}\langle x, y\rangle$ for all $x, y \in \mathcal{X}$, so $\phi(A)$ must be precisely $\mathcal{K}(\mathcal{X})$. Suppose $\phi(a) = 0$ for some a in A. Since ${}_A\langle \mathcal{X}, \mathcal{X}\rangle$ spans a dense ideal, we can approximate a by an element of the form $a \sum_i {}_A\langle x_i, y_i\rangle$. But if $\phi(a) = 0$, then

$$a \sim a \sum_i {}_A\langle x_i, y_i\rangle = \sum_i {}_A\langle a \cdot x_i, y_i\rangle = 0$$

Thus ϕ is an isomorphism. □

Definition 1.10. Let $\mathcal{X}$, $\mathcal{Y}$ be $A-B$-imprimitivity bimodules. An $A-B$-*imprimitivity bimodule isomorphism* is a bimodule isomorphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ that preserves inner products. That is $\langle \varphi(x), \varphi(y)\rangle = \langle x, y\rangle$ for both inner products and all x, $y \in \mathcal{X}$.

Definition 1.11. [8] Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule. A *multiplier* of $\mathcal{X}$ is a pair $m = (m_A, m_B)$, where $m_A\colon A \to \mathcal{X}$ is A-linear, $m_B\colon B \to \mathcal{X}$ is B-linear, and

$$m_A(a) \cdot b = a \cdot m_B(b) \qquad \forall a \in A, b \in B.$$

We write $\mathcal{M}(\mathcal{X})$ for the set of multipliers of $\mathcal{X}$.

As usual, we define $a \cdot m := m_A(a)$ and $m \cdot b := m_B(b)$. Further, there is an injective embedding of $\mathcal{X}$ into $\mathcal{M}(\mathcal{X})$ via $x \mapsto (x_A\colon a \mapsto a \cdot x, x_B\colon b \mapsto x \cdot b)$. $\mathcal{M}(\mathcal{X})$ is an A-B-bimodule, and we refer to it as the *multiplier bimodule* of $\mathcal{X}$. In case $\mathcal{X} = {}_A\mathcal{A}_A$. The multiplier bimodule over $\mathcal{X}$ is equal to the multiplier algebra $\mathcal{M}(A)$.

Proposition 1.12. *[8] Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule. Then $\mathcal{M}(\mathcal{X})$ is an $A-B$ bimodule, satisfying the following conditions:*

(i) $A \cdot \mathcal{M}(\mathcal{X}) \subseteq \mathcal{X}$ *and* $\mathcal{M}(\mathcal{X}) \cdot B \subseteq \mathcal{X}$

(ii) If M is any other $A-B$ bimodule which contains $\mathcal{X}$ and satisfies (i), then there exists a unique bimodule homomorphism $M \to \mathcal{M}(\mathcal{X})$ which is the identity on $\mathcal{X}$.

Moreover, any $A-B$-bimodule which contains $\mathcal{X}$ and satisfies conditions (i) and (ii) is isomorphic (as an $A-B$ bimodule) to $\mathcal{M}(\mathcal{X})$.

Proof. Cf. [8, Proposition 1.2] □

Definition 1.13. Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule. We denote by $\mathcal{L}(B, \mathcal{X})$ the set of all B-linear operators $T\colon B \to \mathcal{X}$ which are adjointable, i.e., for which there is a B-linear map $T^*\colon \mathcal{X} \to B$, such that

$$\langle T(b), x\rangle_B = b^* T^*(x) \qquad \forall b \in B, x \in \mathcal{X}$$

Similarly, we define the set $\mathcal{L}(A, \mathcal{X}$

Lemma 1.14. *Let $m \in \mathcal{M}(\mathcal{X})$. Then $m_B \in \mathcal{L}(B, \mathcal{X})$, with $m_B^*(x)$ given by the unique element of B satisfying*

$$m_A({}_A\langle z, x\rangle) = z \cdot (m_B^*(x))^* \qquad \forall z \in \mathcal{X} \tag{1.1}$$

Similarly, $m_A \in \mathcal{L}_A(A, X)$, with $m_A^*(x)$ characterized by

$$(m_A^*(x))^* \cdot z = m_B(\langle x, z\rangle_B) \qquad \forall z \in \mathcal{X} \tag{1.2}$$

Moreover, $m_B^*(a \cdot x) = \langle m_A(a^*), x\rangle_B$ and $m_A^*(x \cdot b) = {}_A\langle x, m_B(b^*)\rangle$.

Proof. Cf. [8, Lemma 1.3] □

Definition 1.15. [20] Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule, $\mathcal{Y}$ a $B-C$-imprimitivity bimodule. Let $\mathcal{X} \odot \mathcal{Y}$ be the algebraic tensor product. Consider the subspace $N := \operatorname{span}\{(x \cdot b \otimes y - x \otimes b \cdot y) : x \in \mathcal{X},\ y \in \mathcal{Y},\ b \in B\}$ and define

$$\mathcal{X} \odot_B \mathcal{Y} := (\mathcal{X} \odot \mathcal{Y})/N$$

Define some pre-inner products on $\mathcal{X} \odot_B \mathcal{Y}$ by

$$\begin{aligned} \langle\!\langle x_1 \otimes_B y_1, x_2 \otimes_B y_2 \rangle\!\rangle_C &:= \langle \langle x_2, x_1\rangle_B \cdot y_1, y_2\rangle_C \\ {}_A\langle\!\langle x_1 \otimes_B y_1, x_2 \otimes_B y_2 \rangle\!\rangle &:= {}_A\langle x_1, x_2 \cdot {}_B\langle y_2, y_1\rangle\rangle \end{aligned}$$

The completion $\mathcal{X} \otimes_B \mathcal{Y}$ of $\mathcal{X} \odot_B \mathcal{Y}$ with respect to the inner pre-products is called the *internal tensor product* of $\mathcal{X}$ and $\mathcal{Y}$.

Remark 1.16. The inner product defined on $\mathcal{X} \otimes_B \mathcal{Y}$ are not arbitrary. In fact, they are uniquely determined by the condition that $\mathcal{X} \otimes_B \mathcal{Y}$ becomes an $A-C$-imprimitivity bimodule. For further details, see [20]

Remark 1.17. To check that a linear map $\varphi \colon \mathcal{X} \to \mathcal{Y}$ is an isomorphism, it suffices to show that it preserves inner products and that its image is dense in $\mathcal{Y}$.
To see this, note that since φ preserves inner products, hence it is isometric and therefore has closed range. Thus from density of the image we can deduce surjectivity. Also for $x,\ y \in \mathcal{H},\ b \in B$ we have

$$\begin{aligned} \langle \varphi(x), \varphi(y \cdot b)\rangle_B &= \langle x, y \cdot b\rangle_B = \langle x, y\rangle_B \cdot b \\ &= \langle \varphi(x), \varphi(y)\rangle \cdot b = \langle \varphi(x), \varphi(y) \cdot b\rangle \end{aligned}$$

Hence φ commutes with the B-action and is therefore a right B-module isomorphism. An analogue calculation shows that φ is also a left A-module isomorphism.

Definition 1.18. [20] Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule. Let $\mathcal{X}^*$ denote the conjugate vector space of $\mathcal{X}$. Then there is an additive bijection $\flat \colon \mathcal{X} \to \mathcal{X}^*$ such that $\flat(\lambda x) = \overline{\lambda}\flat(x)$. $\mathcal{X}^*$ is a $B-A$-imprimitivity bimodule with module actions

$$\begin{aligned} b \cdot \flat(x) &= \flat(x \cdot b^*) \\ \flat(x) \cdot a &= \flat(a^* \cdot x) \end{aligned}$$

and inner products

$$\begin{aligned} {}_B\langle \flat(x), \flat(y)\rangle &= \langle x, y\rangle_B \\ \langle \flat(x), \flat(y)\rangle_A &= {}_A\langle x, y\rangle \end{aligned}$$

for all y, $x \in \mathcal{X}$, $a \in A$, $b \in B$. The module $\mathcal{X}^*$ is called the *dual* of $\mathcal{X}$

Proposition 1.19. *[20] Let $\mathcal{X}$ be an $A-B$-imprimitivity module, and let $\mathcal{X}^*$ be its dual. Then there is an isomorphism $\varphi_B\colon {}_B(\mathcal{X}^* \otimes_A \mathcal{X})_B \to {}_B\mathcal{B}_B$ such that $\varphi_B(\flat(x) \otimes_A y) = \langle x, y\rangle_B$ for all x, $y \in \mathcal{H}$ and an isomorphism $\varphi_A\colon {}_A(\mathcal{X} \otimes_B \mathcal{X}^*)_A \to {}_A\mathcal{A}_A$ such that $\varphi_A(x \otimes_B \flat(y)) = {}_A\langle x, y\rangle$*

Proof. The map $(\flat(x), y) \mapsto \langle x, y\rangle$ is bilinear and A-balanced, hence gives a linear map $\varphi_B\colon \mathcal{X}^* \otimes_A \mathcal{X} \to B$ satisfying $\varphi_B(\flat(x) \otimes_A y) = \langle x, y\rangle_B$. We prove that φ_B is an isometry by some easy calculations:

$$\begin{aligned} &\langle \varphi_B(\sum_i \flat(x_i) \otimes_A y_i), \varphi_B(\sum_j \flat(u_j) \otimes_A v_j)\rangle_B \\ &\qquad = \sum_{i,j} \langle\langle x_i, y_i\rangle_B, \langle u_j, v_j\rangle_B\rangle_B \\ &\qquad = \sum_{i,j} \langle y_i, x_i\rangle_B \langle u_j, v_j\rangle_B \\ &\qquad = \sum_{i,j} \langle y_i, x_i \cdot \langle u_j, v_j\rangle_B\rangle_B \\ &\qquad = \sum_{i,j} \langle y_i, {}_A\langle x_i, u_j\rangle \cdot v_j\rangle_B \\ &\qquad = \sum_{i,j} \langle\langle \flat(u_j), \flat(x_i)\rangle_A \cdot y_i, v_j\rangle_B \\ &\qquad = \langle\!\langle \sum_i \flat(x_i) \otimes_A y_i, \sum_j \flat(u_j) \otimes_A v_j \rangle\!\rangle_B \end{aligned}$$

An analogue computation shows that φ_A also preserves the left inner product. Since an imprimitivity bimodule is full as a left respectively right Hilbert module, the image of φ_A is dense in B, hence φ_A is an imprimitivity bimodule isomorphism. ☐

Proposition 1.20. *The inverse of an imprimitivity bimodule is determined uniquely up to isomorphisms*

Proof. Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule and $\mathcal{Y}$ be a $B-A$-imprimitivity bimodule such that there is an isomorphism $v\colon \mathcal{X} \otimes_B \mathcal{Y} \to A$, then we get an induced isomorphism

$$\hat{v}\colon \mathcal{Y} \xrightarrow{\sim} B \otimes_B \mathcal{Y} \xrightarrow{\varphi_B^{-1} \otimes_B \mathrm{Id}_{\mathcal{Y}}} (\mathcal{X}^* \otimes_A \mathcal{X}) \otimes_B \mathcal{Y} \xrightarrow{\mathrm{Id}_{\mathcal{X}^*} \otimes_A v} \mathcal{X}^* \otimes_A A \xrightarrow{\sim} \mathcal{X}^*$$

☐

Corollary 1.21. *Let $\mathcal{X}$ be an $A-B$-imprimitivity bimodule, $\mathcal{Y}$ be a $B-A$-imprimitivity bimodule and let $v\colon \mathcal{X} \otimes_B \mathcal{Y} \xrightarrow{\sim} A$ and $w\colon \mathcal{Y} \otimes_A \mathcal{X} \xrightarrow{\sim} B$ be imprimitivity bimodule isomorphisms, and assume that*

$$v \otimes_A \mathrm{Id}_{\mathcal{X}} = \mathrm{Id}_{\mathcal{X}} \otimes_B w\colon \mathcal{X} \otimes_B \mathcal{Y} \otimes_A \mathcal{X} \to \mathcal{X}. \tag{1.3}$$

Then $\hat{w}$ and $\hat{v}$ are inverse to each other.

Proof. Assume that $\mathcal{Y} = \mathcal{X}^*$ and that v and w are the canonical isomorphisms φ_A respectively φ_B as defined in Proposition 1.19. Notice, that condition (1.3) is equivalent to

$$\mathrm{Id}_{\mathcal{Y}} \otimes_A v = w \otimes_B \mathrm{Id}_{\mathcal{Y}} \colon \mathcal{Y} \otimes_A \mathcal{X} \otimes_B \mathcal{Y} \to \mathcal{Y},$$

since

$$\begin{tikzcd} \mathcal{X} \otimes_B \mathcal{Y} \otimes_A \mathcal{X} \otimes_B \mathcal{Y} \arrow[r, bend left, "w\otimes \mathrm{Id}"] \arrow[r, "\mathrm{Id}\otimes w"] \arrow[r, bend right, "w\otimes w"'] & \mathcal{X} \otimes_B \mathcal{Y} \arrow[r, "w"] & A. \end{tikzcd}$$

Then calculate

$$\begin{aligned} \hat{w} \circ \hat{v} \colon \mathcal{X}^* \cong B \otimes_B \mathcal{X}^* &\xrightarrow{\varphi_B^{-1} \otimes_B \mathrm{Id}_{\mathcal{X}^*}} \mathcal{X}^* \otimes_A \mathcal{X} \otimes_B \mathcal{X}^* \\ &\xrightarrow{\mathrm{Id}_{\mathcal{X}^*} \otimes_A \varphi_A} \mathcal{X}^* \otimes_A A \cong B \otimes_B \mathcal{X}^* \\ &\xrightarrow{\varphi_B^{-1} \otimes_B \mathrm{Id}_{\mathcal{X}^*}} \mathcal{X}^* \otimes_A \mathcal{X} \otimes_B \mathcal{X}^* \\ &\xrightarrow{\mathrm{Id}_{\mathcal{X}^*} \otimes_A \varphi_A} \mathcal{X}^* \otimes_A A \cong \mathcal{X}^*. \end{aligned}$$

That is,

$$\begin{aligned} \hat{w} \circ \hat{v} : b \cdot \flat(x) \cdot a \overset{\sim}{\longmapsto} b \otimes_B \flat(x) \cdot a &\xmapsto{\varphi_B^{-1} \otimes_B \mathrm{Id}_{\mathcal{H}^*}} \flat(z_b) \otimes_A y_b \otimes_B \flat(x) \cdot a \\ &\xmapsto{\mathrm{Id}_{\mathcal{H}^*} \otimes_A \varphi_A} \flat(z_b) \otimes_A \langle y_b, \flat(x) \cdot a \rangle_A \\ &\overset{\sim}{\longmapsto} {}_B\langle \flat(z_b), y_b \rangle \otimes_B \flat(x) \cdot a \\ &\xmapsto{\varphi_B^{-1} \otimes_B \mathrm{Id}_{\mathcal{H}^*}} \flat(z_b) \otimes_A y_b \otimes_B \flat(x) \cdot a \\ &\xmapsto{\mathrm{Id}_{\mathcal{H}^*} \otimes_A \varphi_A} \flat(z_b) \otimes_A \langle y_b, \flat(x) \cdot a \rangle_A \\ &\overset{\sim}{\longmapsto} {}_B\langle \flat(z_b), y_B \rangle \cdot x \cdot a \end{aligned}$$

So $\hat{w} \circ \hat{v}$ acts as identity. From the former equation system one can deduce that obviously condition (1.3) holds in case that v and w are the canonical isomorphisms. On the other hand one can deduce that $\hat{w} \circ \hat{v}$ acts as identity for any v, w fulfilling (1.3). □

Remark 1.22. Let $z_1, z_2, x \in \mathcal{X}$, $y \in \mathcal{Y}$. Using the condition $v \otimes_A \mathrm{Id}_{\mathcal{X}} = \mathrm{Id}_{\mathcal{X}} \otimes_B w$ from the former corollary, we can calculate

$$\begin{aligned} {}_B\langle \hat{v}(\langle z_1, z_2 \rangle \cdot y), \flat(x) \rangle &= {}_B\langle \flat(z_1) \cdot v(z_2 \otimes_B y), \flat(x) \rangle \\ &= {}_B\langle \flat(z_1), \flat(x) \cdot v(z_2 \otimes_B y)^* \rangle \\ &= {}_B\langle \flat(z_1), \flat(v(z_2 \otimes_B y) \otimes_A x) \rangle \\ &= \langle z_1, z_2 \otimes_B w(y \otimes_A x) \rangle_B \\ &= \langle z_1, z_2 \rangle_B w(y \otimes_A x) \\ &= w(\langle z_1, z_2 \rangle_B \cdot y \otimes_A x) \end{aligned}$$

Thus, ${}_B\langle \hat{v}(y), \flat(x) \rangle = w(y \otimes_A x)$.

Theorem 1.23 (Rieffel Correspondence). *[6] Let A, B be C^*-algebras, $\mathcal{X}$ be an $A-B$-imprimitivity bimodule and $I \subset B$ be a closed ideal. Define the* ideal induced by I *as*

$$\mathcal{X}\text{-Ind}\, I := \{a \in A : a\mathcal{X} \subset \mathcal{X}I\}$$

Then there is an inclusion preserving one-to-one correspondence between closed ideals in B, induced ideals in A and submodules of the form $\mathcal{X}I$. Moreover, for any closed ideal I in B, we get

(i) $\mathcal{X}\text{-Ind}\, I = \overline{{}_A\langle \mathcal{X}I, \mathcal{X}I\rangle}$

(ii) $(\mathcal{X}\text{-Ind}\, I)\mathcal{X} = \mathcal{X}I$

(iii) $\overline{\langle (\mathcal{X}\text{-Ind}\, I)\mathcal{X}, (\mathcal{X}\text{-Ind}\, I)\mathcal{X}\rangle}_B = I$

1.2 C^*-correspondences

Definition 1.24. Let A, B be C^*-algebras, a *C^*-correspondence* ${}_{\phi(A)}\mathcal{X}_B$ from A to B is a full Hilbert B-module $\mathcal{X}_B$, together with a non-degenerate homomorphism $\phi \colon A \to \mathcal{L}(\mathcal{X}_B)$. We call ϕ the *left action* of A.

Example 1.25. Let A, B be C^*-algebras, $\phi \colon A \to \mathcal{M}(B)$ be a non-degenerate homomorphism, then the Hilbert B-module $\mathcal{B}_B$ becomes a C^*-correspondence ${}_{\phi(A)}\mathcal{B}_B$ since $\mathcal{M}(B) \cong \mathcal{L}(\mathcal{B}_B)$ (cf. [16]).

Remark 1.26. In [5], a *right Hilbert A-B bimodule* is defined as a full Hilbert B-module, with a non-degenerate left action by a C^*-algebra A. Since $\langle a \cdot x, y\rangle = \langle x, a^* \cdot y\rangle$ for any right Hilbert $A-B$ bimodule, A acts by adjointable operators and the left action may be expressed by a homomorphism $\phi \colon A \to \mathcal{L}(\mathcal{E}_B)$ such that $a \cdot x := \phi(a)x$. Therefore there is a one-to-one correspondence between C^*-correspondences and right Hilbert bimodules. We will use this correspondence implicitly on several occasions.

Definition 1.27. Considering the previous remark, an *isomorphism* of C^*-correspondences is a bimodule isomorphism, which is unitary with respect to the inner product

Proposition 1.28. *[5] Let A, C be C^* algebras, and ϕ, ψ be non-degenerate homomorphisms of A into $M(C)$. Then ${}_{\phi(A)}\mathcal{C}_C \cong {}_{\psi(A)}\mathcal{C}_C$ iff there exists $u \in \mathcal{UM}(C)$ such that $u\psi = \phi u$ (we say u* intertwines *ψ and ϕ).*

Proof. If $u\psi = \phi u$, the map $c \mapsto uc$ is a Hilbert module automorphism of $\mathcal{C}_C$ which intertwines the left-actions coming from ϕ and ψ.
Conversely, suppose ${}_{\phi(A)}\mathcal{C}_C \cong {}_{\psi(A)}\mathcal{C}_C$, so there exists a linear bijection $\lambda \colon C \to C$, such that

$$\lambda(cd) = \lambda(c)d, \qquad \lambda(c)^*\lambda(d) = c^*d, \quad \text{and} \quad \lambda(\phi(a)c) = \psi(a)\lambda(c)$$

for each $c, d \in C$. Define $\mu \colon C \to C$ by $\mu(c) = \lambda^{-1}(c^*)^*$. The first two imply that (μ, λ) is an invertible multiplier of C, thus there exists an invertible element $u \in M(C)$, such that $\lambda(c) = u \cdot c$ for all c. Since

$$u^{-1} \cdot c = \lambda^{-1}(c) = \lambda^{-1}(c^{**})^{**} = \mu(c^*)^* = (c^* \cdot u)^* = u^* \cdot c$$

u is unitary, and by the third property of λ, u intertwines ϕ and ψ. ∎

Definition 1.29. Let ${}_{\phi(A)}\mathcal{X}_B$ and ${}_{\psi(B)}\mathcal{Y}_C$ be C^*-correspondences, denote by $\mathcal{X} \odot \mathcal{Y}$ the algebraic tensor product. Consider the subspace $N := \operatorname{span}\{x \cdot b \odot y - x \odot \psi(b)y \colon x \in \mathcal{X}, y \in \mathcal{Y}, b \in B\}$ and define

$$\mathcal{X} \odot_B \mathcal{Y} := (\mathcal{X} \odot \mathcal{Y})/N$$

Let A and C act on $\mathcal{X} \odot_B \mathcal{Y}$ by

$$a \cdot (x \odot_B y) := (\phi(a)x \odot_B y) \qquad (x \odot_B y) \cdot c := x \odot_B (y \cdot c)$$

Further, define some pre-inner product on $\mathcal{X} \odot \mathcal{Y}$ by

$$\langle\!\langle x_1 \odot y_1, x_2 \odot y_2 \rangle\!\rangle_C = \langle y_1, \psi(\langle x_1, x_2\rangle_B) y_2\rangle_C$$

The completion $\mathcal{X} \otimes \mathcal{Y} := \mathcal{X} \odot_B \mathcal{Y}$ is called *internal* or *interior tensor product.*

Proposition 1.30. *The internal tensor product of two C^*-correspondences ${}_{\phi(A)}\mathcal{X}_B$ and ${}_{\psi(B)}\mathcal{Y}_C$ is a C^*-correspondence from A to C.*

Proof. The proof, that $\mathcal{X} \otimes \mathcal{Y}$ is a Hilbert C^*-module, can be found in [16, Proposition 4.5]. To show, that it is indeed a C^*-correspondence, we have to check that the left action is adjointable and non-degenerate. The first part is just some easy calculations:

$$\begin{aligned}
\langle\!\langle a \cdot (x_1 \otimes y_1), x_2 \otimes y_2 \rangle\!\rangle_C &= \langle\!\langle (\phi(a)x_1) \otimes y_1, x_2 \otimes y_2 \rangle\!\rangle_C \\
&= \langle y_1, \psi(\langle \phi(a)x_1, x_2\rangle_B) y_2\rangle_C \\
&= \langle y_1, \psi(\langle x_1, \phi(a)^* x_2\rangle_B) y_2\rangle_C \\
&= \langle\!\langle x_1 \otimes y_1, ((\phi(a)^* x_2) \otimes y_2 \rangle\!\rangle_C \\
&= \langle\!\langle x_1 \otimes y_1, a^* \cdot (x_2 \otimes y_2) \rangle\!\rangle_C
\end{aligned}$$

Further, since $\phi(A)\mathcal{X}$ is dense in $\mathcal{X}$, $\phi(A)(\mathcal{X} \odot_B \mathcal{Y})$ is obviously dense in $\mathcal{X} \odot_B \mathcal{Y}$ and hence $\phi(A)(\mathcal{X}) \otimes_B \mathcal{Y})$ is dense in $\mathcal{X} \otimes_B \mathcal{Y}$. That is, A act non-degenerate. $\square$

Remark 1.31. Let ${}_{\phi(A)}\mathcal{X}_B$ be a C^*-correspondence. Then obviously

$${}_{\phi(A)}\mathcal{X}_B \otimes_B {}_{\mathrm{Id}_B(B)}\mathcal{B}_B \cong {}_{\phi(A)}\mathcal{X}_B \qquad \text{and} \qquad {}_{\mathrm{Id}_A(A)}\mathcal{A}_A \otimes_A {}_{\phi(A)}\mathcal{X}_B \cong {}_{\phi(A)}\mathcal{X}_B$$

Lemma 1.32. *[20] Let $\mathcal{E}_A$ be a Hilbert A-module, define*

$$\begin{aligned} D_x \colon \mathcal{E}_A &\to A \\ y &\mapsto \langle x, y\rangle \end{aligned} \qquad \qquad \begin{aligned} L_x \colon A &\to \mathcal{E}_\mathcal{A} \\ a &\mapsto x \cdot a \end{aligned}$$

then $D_x^* = L_x$ and thus $D_x \in \mathcal{L}(\mathcal{E}_A, A)$ and $L_x \in \mathcal{L}(A, \mathcal{E}_A)$.
Further $D \colon x \mapsto D_x$ defines an isometric conjugate linear isomorphism $\mathcal{E}_A \xrightarrow{\sim} \mathcal{K}(\mathcal{E}_A, A_A)$ and $L \colon x \mapsto L_x$ defines an isometric linear isomorphism $\mathcal{E}_A \xrightarrow{\sim} \mathcal{K}(A_A, \mathcal{E}_A)$.

Proof. [20] To prove, that $D_x^* = L_x$, calculate

$$\langle D_x(y), a\rangle_A = \langle x, y\rangle_A^* a = \langle y, x\rangle a = \langle y, x \cdot a\rangle_A = \langle y, L_x(a)\rangle.$$

D is obviously conjugate linear, because the inner product is conjugate linear in its first argument. To see that D is isometric, calculate

$$\|D_x\| = \sup\{\|\langle x, y\rangle_A\|_A : \|y\|_A \leq 1\} = \|x\|_{\mathcal{E}_A}$$

This yields in particular, that D has closed range. Since $D_{x\cdot a^*} = \theta_{x,a}$, the range of D contains all finite linear combinations of operators of the form $\theta_{x,a}$. Since these are dense in $\mathcal{K}(\mathcal{E}_A, A)$, D is surjective.
For the assertions about L, note that $L\colon x \mapsto D_x^*$ and apply the first part. $\square$

Proposition 1.33. *[6] Let ${}_{\varphi(A)}\mathcal{X}_B$, ${}_{\psi(B)}\mathcal{Y}_A$ be C^*-Correspondences, such that*

$$ {}_{\varphi(A)}\mathcal{X}_B \otimes_B {}_{\psi(B)}\mathcal{Y}_A \cong {}_A A_A \quad \text{and} \quad {}_{\psi(B)}\mathcal{Y}_A \otimes_A {}_{\varphi(A)}\mathcal{X}_B \cong {}_B B_B $$

Then ${}_{\varphi(A)}\mathcal{X}_B$ is an $A-B$-imprimitivity bimodule and $\mathcal{Y} \cong \mathcal{X}^*$

Proof. The second statement is clear since it has already been proved in Theorem 1.20. So we only have to show that ${}_{\varphi(A)}\mathcal{X}_B$ is an imprimitivity bimodule. By proposition 1.9 that means, φ is an isomorphism between A and $\mathcal{K}(\mathcal{X}_B)$.
At first, we show that φ is faithful. Let $\Psi_A\colon {}_{\varphi(A)}(\mathcal{X} \otimes_B \mathcal{Y})_A \xrightarrow{\sim} {}_A A_A$ be the given isomorphism and $L\colon A \to \mathcal{L}(A_A)$, $a \mapsto (L_a\colon b \mapsto ab)$ be the canonical left action.
Assume $\varphi(a)x = 0$ for all $x \in \mathcal{X}$, then $\varphi(a)(x \otimes_B y) = 0$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and thus $\varphi(a)z = 0$ for all $z \in \mathcal{X} \otimes_B \mathcal{Y}$. But since $\Psi_A(\varphi(a)z) = L_a\Psi_A(z) = 0$, $a = 0$ because L is faithful (cf. [16]).
To see that $\varphi(A) = \mathcal{K}(\mathcal{X}_B)$, let $\Psi_B\colon {}_{\psi(B)}(\mathcal{Y} \otimes_A \mathcal{X}) \xrightarrow{\sim} {}_B B_B$, and define $\Phi\colon \mathcal{Y} \to \mathcal{L}(\mathcal{X}, B)$ by

$$\Phi(y)x = \Psi_B(y \otimes x)$$

Then

$$\Phi(\psi(b)y)x = \Psi(\psi(b)y \otimes x) = L_b\Psi(y \otimes x) = L_b\Psi(y)x$$

note that $\mathcal{L}(\mathcal{X}, B)$ becomes a Hilbert $\mathcal{L}(\mathcal{X})$-module with inner product $\langle T_1, T_2\rangle = T_1 T_2^*$.

$$\begin{aligned}
\langle\langle\Phi(y_1), \Phi(y_2)\rangle x_1, x_2\rangle &= \langle\Phi(y_1)^*\Phi(y_2)x_1, x_2\rangle \\
&= \langle\Phi(y_2)x_1, \Phi(y_1)x_2\rangle_B \\
&= \langle\Psi_B(y_2 \otimes x_1), \Psi_B(y_1 \otimes x_2)\rangle \\
&= \langle y_2 \otimes x_1, y_1 \otimes x_2\rangle \\
&= \langle\varphi(\langle y_1, y_2\rangle_A)x_1, x_2\rangle_B
\end{aligned}$$

for all $x_i \in \mathcal{X}$, $y_i \in \mathcal{Y}$. Hence $\langle\Phi(y_1), \Phi(y_2)\rangle = \varphi(\langle y_1, y_2\rangle_A)$ and thus $\Phi(\psi(b)x \cdot a) = L_b\Phi(x) \cdot \varphi(a)$ (cf. [5, Remark 1.17 (2)]). Now

$$\begin{aligned}
\mathcal{X} = \varphi(A)\mathcal{X} = \overline{\Phi(\mathcal{Y})^*\Phi(\mathcal{Y})\mathcal{X}} = \overline{\Phi(\mathcal{Y})^*\Psi(\mathcal{Y} \otimes_A \mathcal{X})} \\
= \overline{\Phi(\mathcal{Y})^* B} = \Phi(\psi(B)\mathcal{Y})^* = \Phi(\mathcal{Y})^*
\end{aligned}$$

and using Lemma 1.32

$$\varphi(A) = \overline{\Phi(\mathcal{Y})^*\Phi(\mathcal{Y})} = \overline{\mathcal{X}\mathcal{X}^*} = \mathcal{K}(\mathcal{X})$$

$\square$

1.3 2-Categories and related structures

Definition 1.34. A (*strict*) *2-category* $\mathfrak{C}$ consist of:

(i) a collection of objects $\mathfrak{C}^{(0)}$

(ii) a collection of arrows between objects $\mathfrak{C}^{(1)}$ together with a source map $s\colon \mathfrak{C}^{(1)} \to \mathfrak{C}^{(0)}$, a target map $r\colon \mathfrak{C}^{(1)} \to \mathfrak{C}^{(0)}$, a multiplication

$$\begin{array}{rcl} \circ\colon \mathfrak{C}^{(1)}{}_s\times_r \mathfrak{C}^{(1)} & \to & \mathfrak{C}^{(1)} \\ (f,g) & \mapsto & f\circ g \end{array}$$

such that for any $f,g \in \mathfrak{C}^{(1)}{}_s\times_r \mathfrak{C}^{(1)}$, $s(f\circ g) = s(g)$ and $r(f\circ g) = r(f)$, and for any composable $f,g,h \in \mathfrak{C}^{(1)}$, $f\circ(g\circ h) = (f\circ g)\circ h$, and a unitarrow $1_x \in \mathfrak{C}^{(1)}$ for all $x \in \mathfrak{C}^{(0)}$, such that, $1_x \circ g = g$ respectively $f\circ 1_x = f$ for all $f,g\in\mathfrak{C}^{(1)}$ satisfying $r(g) = x$ respectively $s(f) = x$.
We define $\mathfrak{C}^{(1)}(y,x) := s^{-1}(y)\cap r^{-1}(x)$ to be the collection of arrows with source y and target x.

(iii) a collection of bigons between arrows $\mathfrak{C}^{(2)}$, together with a source map $s^{(2)}\colon \mathfrak{C}^{(2)} \to \mathfrak{C}^{(1)}$ and a target map $r^{(2)}\colon \mathfrak{C}^{(2)} \to \mathfrak{C}^{(1)}$, an associative *horizontal multiplication* $\cdot_h\colon \mathfrak{C}^{(2)}\times\mathfrak{C}^{(2)} \to \mathfrak{C}^{(2)}$ and an associative *vertical multiplication* $\circ\colon \mathfrak{C}^{(2)}\times\mathfrak{C}^{(2)}\to\mathfrak{C}^{(2)}$. Such that for any $(f_1,f_2),(g_1,g_2)\in \mathfrak{C}^{(1)}{}_s\times_r\mathfrak{C}^{(1)}$ satisfying $a(f_1) = g_1$ and $b(f_2) = g_2$ for some bigons a,b

$$a(f_1)\circ b(f_2) = (a\cdot_h b)(f_1\circ f_2),$$

and for any $f,g,h \in \mathfrak{C}(x,y)$ such that $a(f) = g$ and $b(g) = h$ for some bigons a,b

$$b(a(f)) = (b\circ a)(f).$$

Further, horizontal and vertical multiplication have to respect some *coherence law*, that is: let a_1,a_2,b_1,b_2 be some bigons in $\mathfrak{C}$, such that $s^{(2)}(b_i) = r^{(2)}(a_i)$ for $i = 1,2$ and $a_2\cdot_h a_1$ and $b_1\cdot_h b_2$ are defined. Then

$$(b_1\cdot_h b_2)\circ(a_1\cdot_h a_2) = (b_1\circ a_1)\cdot_h(b_2\circ a_2)$$

Remark 1.35. For the sake of convenience, it is common practice to omit the $\circ$ to denote multiplication of arrows or vertical multiplication of bigons. That is, $f\circ g$ becomes fg and $a\circ b$ becomes ab for some composable arrows $f,g\in\mathfrak{C}^{(1)}$ respectively some vertically composable bigons $a,b\in\mathfrak{C}^{(2)}$.

Remark 1.36. A strict 2-category $\mathfrak{C}$ can be defined through commutative diagramms. Let x,y,z denote objects in $\mathfrak{C}$, $\to$ arrows and $\Rightarrow$ bigons. Then we get the multiplication between arrows by:

$$\begin{tikzcd} x & y \arrow[l, bend right] & z \arrow[l, bend right] & \mapsto & x & z \arrow[l, bend right] \end{tikzcd}$$

The horizontal multiplication will be defined by

$$\begin{tikzcd} z & y \arrow[l, bend right, "f_1"'] \arrow[l, bend left, "g_1"] \arrow[l, phantom, "\Downarrow a"] & x \arrow[l, bend right, "f_2"'] \arrow[l, bend left, "g_2"] \arrow[l, phantom, "\Downarrow b"] & \mapsto & z & x. \arrow[l, bend right, "f_1 f_2"'] \arrow[l, bend left, "g_1 g_2"] \arrow[l, phantom, "\Downarrow a\cdot_h b"] \end{tikzcd}$$

and the vertical multiplication becomes

$$\begin{tikzcd}[column sep=large] y & x \arrow[l, bend right=60, "f"', ""{name=F}] \arrow[l, "g" description, ""{name=G}] \arrow[l, bend left=60, "h", ""{name=H}] \arrow[from=F, to=G, phantom, "\Downarrow a"] \arrow[from=G, to=H, phantom, "\Downarrow b"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}[column sep=large] y & x \arrow[l, bend right=50, "f"', ""{name=F}] \arrow[l, bend left=50, "h", ""{name=H}] \arrow[from=F, to=H, phantom, "\Downarrow b\circ a"] \end{tikzcd}.$$

Further, we can represent the interchange law by

$$\begin{tikzcd}[column sep=large] z & y \arrow[l, bend right=60, "f_1"', ""{name=F1}] \arrow[l, "g_1" description, ""{name=G1}] \arrow[l, bend left=60, "h_1", ""{name=H1}] & x \arrow[l, bend right=60, "f_2"', ""{name=F2}] \arrow[l, "g_2" description, ""{name=G2}] \arrow[l, bend left=60, "h_2", ""{name=H2}] \arrow[from=F1, to=G1, phantom, "\Downarrow a_1"] \arrow[from=G1, to=H1, phantom, "\Downarrow b_1"] \arrow[from=F2, to=G2, phantom, "\Downarrow a_2"] \arrow[from=G2, to=H2, phantom, "\Downarrow b_2"] \end{tikzcd},$$

in the sense that it does not matter if we first compose vertically, and then horizontally or the other way round to get the bigon $f_1 \circ f_2 \Rightarrow h_1 \circ h_2$.

Definition 1.37. A *strict 2-Groupoid* is a 2-Category $\mathcal{G}$ whose object class is a set and where all 1- and 2-morphisms are invertible.
A *strict 2-group* is a 2-groupoid $\mathcal{G}$ whose object class is the one-elemental set denoted by $\{\star\}$.

Definition 1.38. A *weak 2-Category* $\mathfrak{C}$ is similar to a (strict) 2-category, but weakened in the sense, that the unit and associativity law only hold up to isomorphisms. That is, for any composable $f, g, h \in \mathfrak{C}$, there is an invertible bigon $a_{f,g,h} \in \mathfrak{C}^{(2)}$, the *associator*, such that

$$f \circ (g \circ h) \xRightarrow{a_{f,g,h}} (f \circ g) \circ h$$

and for any $x, y \in \mathfrak{C}^{(0)}$, $f \in \mathfrak{C}^{(1)}(x, y)$, there are invertible bigons l_f, r_f, the left and right *unitor*, such that

$$1_x \circ f \overset{l_f}{\Rightarrow} f \overset{r_f}{\Leftarrow} f \circ 1_x.$$

Further the associators and unitors have to respect certain coherence laws. That is, for any composable $e, f, g, h \in \mathfrak{C}^{(1)}$ the following diagrams have to commute

$$\begin{tikzcd} ((e \circ f) \circ g) \circ h \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (e \circ f) \circ (g \circ h) \arrow[r, Rightarrow] & e \circ (f \circ (g \circ h)) \\ (e \circ (f \circ g)) \circ h \arrow[rr, Rightarrow] & & e \circ ((f \circ g) \circ h) \arrow[u, Rightarrow] \end{tikzcd}, \tag{1.4}$$

and provided, that $f \in \mathfrak{C}^{(1)}(y, z)$ and $g \in \mathfrak{C}^{(1)}(x, y)$:

$$\begin{tikzcd} (f \circ 1_y) \circ g \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & f \circ (1_y \circ g) \arrow[dl, Rightarrow] \\ & f \circ g & \end{tikzcd}. \tag{1.5}$$

Definition 1.39. [2] Let $\mathfrak{C}$, $\mathfrak{G}$ be two weak 2-categories. A *morphism of 2-categories* (or *weak 2-functor*) consists of the following data:

- a map $F\colon \mathfrak{C}^{(0)} \to \mathfrak{G}^{(0)}$,
- a Functor $F(x,y)\colon \mathfrak{C}^{(1)}(x,y) \to \mathfrak{G}^{(1)}(F(x),F(y))$ for each pair of objects x, y,
- a natural bigon $\omega(f_1,f_2)\colon F(f_1)\circ F(f_2) \Rightarrow F(f_1\circ f_2)$ for each pair (f_1,f_2) of composable arrows; where naturality means that the following diagrams commute for all pairs of bigons $n_1\colon f_1 \Rightarrow g_1$, $n_2\colon f_2 \Rightarrow g_2$ between composable arrows:

$$\begin{tikzcd} F(f_1)\circ F(f_2) \arrow[r, Rightarrow, "\omega(f_1,f_2)"] \arrow[d, Rightarrow, "F(n_1)\cdot_h F(n_2)"'] & F(f_1\circ f_2) \arrow[d, Rightarrow, "F(n_1\cdot_h n_2)"] \\ F(g_1)\circ F(g_2) \arrow[r, Rightarrow, "\omega(g_1,g_2)"'] & F(g_1\circ g_2); \end{tikzcd} \tag{1.6}$$

- bigons $u_x\colon 1_{F(x)} \Rightarrow F(1_x)$.

As well as some coherence laws

$$\begin{tikzcd} (F(g_1)\circ F(g_2))\circ F(g_3) \arrow[r, Rightarrow, "a'(F(g_1),F(g_2),F(g_3))"] \arrow[d, Rightarrow, "\omega(g_1,g_2)\cdot_h \mathrm{Id}_{F(g_3)}"'] & F(g_1)\circ(F(g_2)\circ F(g_3)) \arrow[d, Rightarrow, "\mathrm{Id}_{F(g_1)}\cdot_h \omega(g_2,g_3)"] \\ F(g_1\circ g_2)F(g_3) \arrow[d, Rightarrow, "\omega(g_1\circ g_2,g_3)"'] & F(g_1)F(g_2\circ g_3) \arrow[d, Rightarrow, "\omega(g_1,g_2\circ g_3)"] \\ F((g_1\circ g_2)\circ g_3) \arrow[r, Rightarrow, "F(a(g_1,g_2,g_3))"'] & F(g_1\circ(g_2\circ g_3)) \end{tikzcd} \tag{1.7}$$

$$\begin{tikzcd} F(1_x)\circ F(g) \arrow[r, Rightarrow, "\omega(1_x,g)"] & F(1_x\circ g) \arrow[d, Rightarrow, "F(l_g)"] \\ 1_{F(x)}\circ F(g) \arrow[u, Rightarrow, "u\cdot_h \mathrm{Id}_{F(g)}"] \arrow[r, Rightarrow, "l_{F(g)}"'] & F(g) \end{tikzcd} \qquad \begin{tikzcd} F(g)\circ F(1_y) \arrow[r, Rightarrow, "\omega(g,1_y)"] & F(g\circ 1_y) \arrow[d, Rightarrow, "F(r_g)"] \\ F(g)\circ 1_{F(y)} \arrow[u, Rightarrow, "\mathrm{Id}_{F(g)}\cdot_h u"] \arrow[r, Rightarrow, "r_{F(g)}"'] & F(g). \end{tikzcd} \tag{1.8}$$

If in addition, the bigons u_x and $\omega(g,f)$ are invertible, we speak of a *homomorphism* of weak 2-categories. If they are all identities, we get a (*strict*) *2-functor* (this notion is only interesting if both 2-categories are strict).

Definition 1.40. A *strict 2-group action* of a 2-group $\mathcal{G}$ on a C^*-algebra A is a (strict) 2-functor from the 2-group into a (strict) category of C^*-algebras, which maps the object $\star \in \mathcal{G}^{(0)}$ onto A.

Remark 1.41. Due to the strictness condition, there is only one reasonable choice for the 2-category of C^*-algebras. That is the category $\mathfrak{C}^*(2)$, which consists of C^*-algebras as objects, non-degenerate $*$-homomorphisms $A \to \mathcal{M}(B)$ as arrows, and unitary intertwiner between such $*$-homomorphisms as bigons. For further details, cf. e.g. [2]

Definition 1.42. [2] Let G be a discrete group, $\mathfrak{C}$ a weak 2-category. We can view G as a weak 2-category $\mathfrak{G}$ by defining $\mathfrak{G}^{(0)} := \{\star\}$, $\mathfrak{G}^{(1)} := G$, and any bigon to be the identity. A *weak action* of G on an object A in $\mathfrak{C}$ is a homomorphism of weak 2-categories from $\mathfrak{G}$ to $\mathfrak{C}$, which maps $\star$ to A. By Definition 1.39, we get

- an arrow $\alpha_g \colon A \to A$ for all $g \in G$
- a bigon $u \colon 1_A \to \alpha_1$
- bigons $\omega(g_1, g_2) \colon \alpha_{g_1}\alpha_{g_2} \Rightarrow \alpha_{g_1 g_2}$.

and the simplified coherence laws

$$\begin{tikzcd}
\alpha_1 \circ \alpha_t \arrow[r, Rightarrow, "\omega(1,g)"] & \alpha_{1\cdot g} \arrow[d, equal] & \alpha_g \circ \alpha_1 \arrow[r, Rightarrow, "\omega(g,1)"] & \alpha_{g\cdot 1} \arrow[d, equal] \\
1 \circ \alpha_g \arrow[u, Rightarrow, "u\cdot_h \mathrm{Id}_{\alpha_g}"] \arrow[r, Leftrightarrow] & \alpha_g & \alpha_g \circ 1 \arrow[u, Rightarrow, "\mathrm{Id}_{\alpha_g}\cdot_h u"] \arrow[r, Leftrightarrow] & \alpha_g
\end{tikzcd} \tag{1.9}$$

and

$$\begin{tikzcd}
(\alpha_{g_1}\alpha_{g_2})\alpha_{g_3} \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow, "\omega(g_1,g_2)\cdot_h \mathrm{Id}_{\alpha_{g_3}}"'] & & \alpha_{g_1}(\alpha_{g_2}\alpha_{g_3}) \arrow[d, Rightarrow, "\mathrm{Id}_{\alpha_{g_1}}\cdot_h \omega(g_2,g_3)"] \\
\alpha_{g_1g_2}\alpha_{g_3} \arrow[dr, Rightarrow, "\omega(g_1g_2,g_3)"'] & & \alpha_{g_1}\alpha_{g_2g_3} \arrow[dl, Rightarrow, "\omega(g_1,g_2g_3)"] \\
& \alpha_{g_1g_2g_3} &
\end{tikzcd} \tag{1.10}$$

where the $\Leftrightarrow$ denote the unit and associativity bigons from the 2-category $\mathfrak{C}$.

Proposition 1.43. *There is a weak 2-category $\mathfrak{Corr}(2)$, which consists of C^*-algebras as object class, C^*-correspondences as 1-morphisms and isomorphisms of C^*-correspondences as 2-morphisms. The composition of 1-morphisms is given by the interior tensor product, that is, for two C^*-correspondences ${}_{\phi(A)}\mathcal{X}_B$, ${}_{\psi(B)}\mathcal{Y}_C$, we define*

$$\mathcal{Y} \circ \mathcal{X} := \mathcal{X} \otimes_B \mathcal{Y}. \tag{1.11}$$

The vertical multiplication of two 2-morphisms b_1, b_2 is given by concatenation, that is

$$b_1 \circ b_2 := b_1 b_2, \tag{1.12}$$

and the vertical multiplication of some bigons a_1, a_2 is given by

$$(a_2 \cdot_h a_1) := a_1 \otimes_B a_2, \tag{1.13}$$

such that for any $x \in \mathcal{X}$, $y \in \mathcal{Y}$,

$$(a_1 \otimes_B a_2)(x \otimes_B y) := a_1(x) \otimes_B a_2(y). \tag{1.14}$$

Proof. Let ${}_{\phi(A)}\mathcal{X}_B$, ${}_{\psi(B)}\mathcal{Y}_C$, ${}_{\theta(C)}\mathcal{Z}_D$ be C^*-correspondences, define the range and source maps by $s({}_{\phi(A)}\mathcal{X}_B) := A$, $r({}_{\phi(A)}\mathcal{X}_B) := B$. Then our multiplication yields

$$\begin{aligned}
s(\mathcal{Y} \circ \mathcal{X}) &= s(\mathcal{X} \otimes_B \mathcal{Y}) = A = s(\mathcal{X}) \\
r(\mathcal{Y} \circ \mathcal{X}) &= r(\mathcal{X} \otimes_B \mathcal{Y}) = C = r(\mathcal{Y})
\end{aligned}$$

Further, by Remark 1.31, for any C^*-algebra A the C^*-correspondence ${}_{\mathrm{Id}_A(A)}\mathcal{A}_A$ fulfills the conditions of a unit arrow, hence we define $1_A := {}_{\mathrm{Id}_A(A)}\mathcal{A}_A$ and the isomorphisms in Remark 1.31 to be the bigons l_A and r_A. Moreover, $r^{(2)} := \mathrm{Ran}$, and $s^{(2)} := \mathrm{Dom}$. To show that $(\mathcal{X} \otimes_B \mathcal{Y}) \otimes_C \mathcal{Z} \cong \mathcal{X} \otimes_B (\mathcal{Y} \otimes_C \mathcal{Z})$, we use the natural isomorphism $(\mathcal{X} \odot_B \mathcal{Y}) \odot_C \mathcal{Z} \xrightarrow{\sim} \mathcal{X} \odot_B (\mathcal{Y} \odot_C \mathcal{Z})$ and show that it respects inner products and module actions.
Since $\langle\langle \cdot, \cdot \rangle\rangle_D$ is an inner product, and hence bilinear. It is sufficient to proof the invariance for some basis vectors $x_i \otimes (y_i \otimes z_i) \in \mathcal{X} \otimes_B (\mathcal{Y} \otimes_C \mathcal{Z})$, $i = 1, 2$. Hence

$$\begin{aligned}
\langle\langle x_1 \otimes (y_1 \otimes z_1), x_2 \otimes (y_2 \otimes z_2) \rangle\rangle_D &= \langle\langle y_1 \otimes z_1, \psi(\langle x_1, x_2 \rangle_B)(y_2 \otimes z_2) \rangle\rangle_D \\
&= \langle z_1, \theta(\langle y_1, \psi(\langle x_1, x_2 \rangle_B) \rangle_C) z_2 \rangle_D \\
&= \langle z_1, \theta(\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle_C) \rangle_D \\
&= \langle\langle (x_1 \otimes y_1) \otimes z_1, (x_2 \otimes y_2) \otimes z_2 \rangle\rangle_D
\end{aligned}$$

Further, we get

$$\begin{aligned}
a \cdot ((x \otimes y) \otimes z) = (a \cdot (x \otimes y)) \otimes z &= ((\phi(a)x) \otimes y) \otimes z \\
&\cong (\phi(a)x) \otimes (y \otimes z) = a \cdot (x \otimes (y \otimes z))
\end{aligned}$$

and an analogous computation for the right action. Thus our isomorphism respects the inner product and module actions, and is therefore an isomorphism of C^*-correspondences, ergo an associator.
The computations to check the diagrams (1.4) and (1.5) of Definition 1.38 are easy, but lengthy, and will therefore be omitted.
Last thing to check is the interchange law. Let $a_i, b_i \in \mathfrak{Corr}(2)^{(2)}$, such that $s^{(2)}(b_i) = r^{(2)}(a_i)$ for $i = 1, 2$ and $a_2 \cdot_h a_1$ and $b_1 \cdot_h b_2$ are defined. Then

$$\begin{aligned}
((b_1 \cdot_h b_2) \circ (a_1 \cdot_h a_2)) &= ((b_2 \otimes b_1) \circ (a_2 \otimes a_1)) \\
&= (b_2 \otimes b_1)(a_2 \otimes a_1) \\
&= b_2 a_2 \otimes b_1 a_1 \\
&= b_1 a_1 \cdot_h b_2 a_2 \\
&= (b_1 \circ a_1) \cdot_h (b_2 \circ a_2).
\end{aligned}$$

$\square$

Chapter 2

Weak Group Actions in $\mathfrak{Corr}(2)$

We are starting with a short wrap-up of some basic properties of locally compact groups, with a focus on separation properties. It follows a more substantial discussion of general bundle theory, concentrating especially on Banach and Banach $*$-algebraic bundles and leading to the definition of a Fell bundle. In the following I will present a result by Buss, Meyer and Zhu, originally stated in [2], which says that weak group actions of discrete groups in $\mathfrak{Corr}(2)$ correspond one-to-one to saturated Fell bundles.
Finally, we will introduce $C_0(X)$-*Algebras*, and prove that they are equivalent to upper semi-continuous C^*-bundles. Using this result, it is easy to derive an equivalence of upper semi-continuous bundles of A-A-imprimitivity bimodules and a $C_0(X,A) - C_0(X,A)$-imprimitivity bimodules. We use this to find a notion of continuity for group actions in $\mathfrak{Corr}(2)$ and show, that this yields an equivalence between continuous group actions in $\mathfrak{Corr}(2)$ and continuous saturated Fell bundles.

2.1 Topological preliminaries

The following lemma is a slight generalization of [23, Proposition 5.3.4.].

Lemma 2.1. *Let X, Y be locally compact spaces, B be a Banach space. Let $f \in C_0(X, C_0(Y,B))$ and define $g(x,y) := f(x)(y)$, then $g \in C_0(X \times Y, B)$.*

Proof. We have to prove that g is continuous at an arbitrary point $(x,y) \in X \times Y$. Given $\varepsilon > 0$, there exists a neighborhood U of x, such that the condition $x' \in U$ implies

$$\|f(x) - f(x')\|_{C_0(Y,B))} = \sup_{y' \in Y} \|g(x,y') - g(x',y')\|_B < \frac{\varepsilon}{2}.$$

There also exists a neighborhood V of y, such that $y' \in V$ implies $\|f(x)(y) - f(x)(y')\|_B < \frac{\varepsilon}{2}$. Consequently, if $(x',y') \in U \times V$, then

$$\|g(x',y') - g(x,y)\|_B \leq \|g(x',y') - g(x,y')\|_B + \|g(x,y') - g(x,y)\|_B < \varepsilon.$$

Hence g is continuous, which proves the assumption. $\square$

Lemma 2.2. *[24, Lemma 1.25] Let X be a locally compact Hausdorff space, then X is completely regular. That is, for every point x in X, and every closed subset $Y \subset X$ such that $x \notin Y$, there is a continuous function $f\colon X \to [0,1]$, such that $f(x) = 1$ and $f(Y) = 0$.*

Definition 2.3. A *topological group* is a group G together with a Hausdorff topology $\mathcal{T}$, such that $(g,h) \mapsto gh^{-1}$ is continuous on $G \times G$ to G.

Remark 2.4. Some authors do not require the topology $\mathcal{T}$ to be Hausdorff. But since we need Hausdorff for many of our results, and since it is common practice to define topological groups to be Hausdorff, we adopt this convention here.

Lemma 2.5. *[24, Lemma 6.2] Let G be a topological group, H be a subgroup of G. Then*

(i) the quotient map $\pi\colon G \to G/H$ is open.

(ii) If H is normal, then G/H is a topological group

Lemma 2.6. *[13, Theorem 8.4] Let G be a topological group, if G is T_0, then G is completely regular*

Definition 2.7. [25] A *locally compact group* is a topological group G, in which every point has a compact neighbourhood.

Remark 2.8. Since any locally compact group is also a locally compact Hausdorff space, by Lemma 2.2 any locally compact group is completely regular.

Proposition 2.9. *[24, Proposition 6.6, Theorem 6.7] Let G be a locally compact group, H be a closed normal subgroup of G. Then G/H is a locally compact group*

2.2 Banach- and Banach algebraic Bundle Theory

Definition 2.10. [11] Let X be a topological Hausdorff space. A *bundle* over X is a pair (B,π), where B is a topological Hausdorff space and $\pi\colon B \to X$ is a continuous, open surjection. We denote the pair (B,π) by $\mathfrak{B}$ and call X the *base space*, B the *bundle space*, and π the *bundle projection*. For every x in X, $\pi^{-1}(x)$ is called *fiber* over x and will be denoted by B_x.

Definition 2.11. [11] Let $\mathfrak{B} = (B,\pi)$ be a bundle, a map $f\colon X \to A$ which satisfies $f(x) \in A_x$ for all $x \in X$ (i.e., $f \in \prod_{x\in X} A_x$) is called a *cross-section*. By $\Gamma(\mathfrak{B}) \subset \prod_{x\in X} A_x$ we denote the the set of continuous cross-sections, and by $\Gamma_0(\mathfrak{B}) \subset \Gamma(\mathfrak{B})$ be the subset of continuous cross-sections vanishing at infinity. Moreover, if for all b in B, there exists a cross-section γ, such that $\gamma(\pi(b)) = b$, we say that $\mathfrak{B}$ *has enough continuous cross-sections.*

Remark 2.12. In his early work (e.g. [10]), Fell used the term *cross-sectional function* for non-continuous cross-section, and the term *cross-section* exclusively in the continuous case. Moreover, in many papers cross-sections are simply referred to as *sections*, and non-continuous cross-sections in this context are often defined as *selections* (e.g. [4]).

Definition 2.13. [11] A *Banach bundle* $\mathfrak{B}$ over a Hausdorff topological space X is a bundle (B, π) together with a binary operation $+$, a norm, and a scalar multiplication such that every fiber B_x is a Banach space and the following conditions hold:

(i) $B \ni b \mapsto \|b\| \in \mathbb{R}_+$ is continuous,

(ii) the operation
$$+ \colon B \times_X B \to B$$
is continuous,

(iii) for every $\lambda \in \mathbb{C}$, $b \in B$, the map $b \mapsto \lambda b$ is continuous,

(iv) if $x \in X$, $\{b_i\}$ net in B, such that $\|b_i\| \to 0$ and $\pi(b_i) \to x \in X$, then $b_i \to 0_x \in B$, where 0_x denotes the zero of the fiber b_x.

Remark 2.14. [25] If we weaken condition (i) to

(i′) $B \ni b \mapsto \|b\| \in \mathbb{R}_+$ is upper semi-continuous,

we call $\mathfrak{B}$ an *upper semi-continuous Banach bundle.* Moreover, any bundle that has a Banach bundle structure admits an upper semi-continuous version.

Remark 2.15. Let $\mathfrak{B}$ be a Banach bundle, then $\Gamma_0(\mathfrak{B})$ is a complex, linear space under pointwise addition, which is closed under multiplication by $C_0(x)$ (cf. [11, II.13.14]). Moreover, if we equip $\Gamma_0(\mathfrak{B})$ with the supremumsnorm, it is even a Banach space, called the C_0 *cross-sectional Banach space.*

Example 2.16. [11] Let A be a Banach space, X a Hausdorff topological space, put $B = A \times X$ and $\pi(a, x) = x$ and equip each fiber $B_x = \pi^{-1}(x)$ with the Banach space structure derived from A via the bijection $a \mapsto (a, x)$. Then $\mathfrak{B} = (B, \pi)$ is a Banach bundle, called the *trivial Banach bundle* with *constant fiber* A. Note that in this case, $\Gamma_0(\mathfrak{B}) = C_0(X, A)$.

Proposition 2.17. *[11, Proposition II.13.15] Let $\mathfrak{B}$ be a Banach bundle, suppose that for each b in B, $\varepsilon > 0$, there exists an $\gamma \in \Gamma(\mathfrak{B})$, such that $\|\gamma(\pi(b)) - b\| < \varepsilon$. Then $\mathfrak{B}$ has enough continuous cross-sections.*

Proposition 2.18. *[10] Let $\mathfrak{B} = (b, \pi)$ be a Banach bundle over X, suppose that $(b_i)_{i \in I}$ is a net in B, such that $\pi(b_i) \to \pi(b)$ in X. Suppose that for each $\varepsilon > 0$, we can find a net $(a_i)_{i \in I}$, indexed by the same I, and an element a in B, such that*

(i) $a_i \to a$ in B

(ii) $\pi(a_i) = \pi(b_i)$ for all i

(iii) $\|b - a\| < \varepsilon$

(iv) $\|b_i - a_i\| < \varepsilon$ for all large enough i

then $b_i \to b$ in B

Proof. Since we could replace $(b_i)_{i\in I}$ by any subnet of itself, it is enough to show that some subnet of $(b_i)_{i\in I}$ converges to b.
Since π is open, we can pass to a subnet, and find for each i an element c_i of B, such that $\pi(c_i) = \pi(b_i)$ and $c_i \to b$. Now, given $\varepsilon > 0$, choose $(a_i)_{i\in I}$ and a as in the hypothesis. Since $c_i \to b$, we have by 2.13 (ii) $c_i - a_i \to b - a$, hence by 2.13 (i) $\|c_i - a_i\| \to \|b - a\| < \varepsilon$, so $\|c_i - a_i\| < \varepsilon$ for large i. Combining this with assumption (iv), we get $\|c_i - b_i\| < 2\varepsilon$ for large enough i. By the arbitrariness of ε, this shows that $\|c_i - b_i\| \to 0$, hence by 2.13 (iv) $c_i - b_i \to 0_{\pi(b)}$. Since $c_i \to b$, it follows from this and 2.13 (ii) that $b_i \to b$ □

Proposition 2.19. *[10] Let A be an untopologized set, X a Hausdorff topological space, and $\pi\colon A \to X$ a surjection, such that for each x in X, the set $A_x = \pi^{-1}(x)$ is a Banach space (with Banach space structure $\cdot$, $+$, $\|\cdot\|$). Let Γ be a complex vector space of cross-sections for (A, π), such that*

(i) for each ξ in Γ, the function $x \mapsto \|\xi(x)\|$ is continuous on X, and

(ii) for each x in X, $\{\xi(x) : \xi \in \Gamma\}$ is dense in A_x.

Then there is a unique topology for A, making (A, π) a Banach bundle such that all elements of Γ are continuous cross-sections for (A, π).

Proof. To prove the existence, let S be the family of all subsets of A of the form

$$W(f, u, \varepsilon) = \{s \in A : \pi(s) \in U, \|s - f(\pi(s))\| < \varepsilon\},$$

where $f \in \Gamma$, U is an open subset of X and $\varepsilon > 0$.
The computations to show, that the given topology indeed has the required properties, are easy, but lengthy and will therefore be omitted.
For the uniqueness: given such a topology on A and a net $(a_i)_{i\in I}$ of elements in A. By Proposition 2.18, $a_i \to a$ iff

(i) $\pi(a_i) \to \pi(a)$ in X,

(ii) for each ξ in Γ, $\|a_i - \xi(\pi(a_i))\| \to \|\pi(a) - \xi(\pi(a))\|$

thus the required topology of A, if it exists, is unique. □

Remark 2.20. Proposition 2.18 and 2.19 still hold if we work with upper semi-continuous Banach bundles. The proves are generally identical and may be found in [25, Theorem C.20, Theorem C.25]. Even though they are stated for C^*-bundles (which we will define later), they use none of the exclusive C^*-properties and can be copied for Banach bundles without any changes.

Definition 2.21. [12] A *Banach algebraic bundle* over a topological group G is a Banach bundle $\mathfrak{B} = (B, \pi)$ over G, together with a binary operation $\cdot$ satisfying

(i) $\pi(b \cdot c) = \pi(b)\pi(c)$

(ii) for each pair $g, h \in G$ the product $\cdot$ is bilinear on $B_g \times B_h$ to B_{gh}

(iii) the product $\cdot$ on B is associative

(iv) $\|b \cdot c\| \le \|b\|\|c\|$

(v) the map $\cdot$ is continuous on $B \times B$ to B

$\mathfrak{B}$ is called *saturated*, if $B_g \cdot B_h$ is dense in B_{gh} for all g, h in G.

Remark 2.22. Condition (i) is equivalent to

(i′) $B_g \cdot B_h \subset B_{gh}$

Remark 2.23. Note that, contrary to the Banach space case, condition (iv) does not imply condition (v).

Lemma 2.24. *[12] Let Γ be a family of continuous cross-sections of $\mathfrak{B}$ such that $\{\Gamma(g) : \gamma \in \Gamma\}$ is dense in B_g for every g in G. Then condition (v) is equivalent to*

(v′) for each pair of elements β, γ in Γ, the map $(g, h) \mapsto \beta(g)\gamma(h)$ is continuous on $G \times G$ to B.

Proof. Evidently, (v)$\Rightarrow$(v′).
For the converse, assume (i) – (iv) and (v′), and let $b_i \to b$ and $c_i \to c$ in B. We have to show, that $b_i c_i \to bc$. To do this, let $\varepsilon > 0$ and let β, γ in Γ, such that

$$\begin{aligned} &\|\beta(\pi(b)) - b\| < \varepsilon(4\|c\|)^{-1} \\ &\|\gamma(\pi(c)) - c\| < \varepsilon(4\|\beta(\pi(b))\|)^{-1} \end{aligned} \tag{2.1}$$

by (v′)

$$\beta(\pi(b_i))\gamma(\pi(c_i)) \to \beta(\pi(b))\gamma(\pi(c)). \tag{2.2}$$

by (i) and 2.17 there is a continuous cross-section α of $\mathfrak{B}$, such that

$$\alpha(\pi(bc)) = \beta(\pi(b))\gamma(\pi(c))$$

from this and (2.2) we obtain

$$\|\beta(\pi(b_i))\gamma(\pi(c_i) - \alpha(\pi(b_i c_i))\| \to 0$$

now (2.1) implies that, for large i,

$$\begin{aligned} &\|\beta(\pi(b_i)) - b_i\| < \varepsilon(4\|c_i\|)^{-1} \\ &\|\gamma(\pi(c_i)) - c_i\| < \varepsilon(4\|\beta(\pi(b_i))\|)^{-1} \end{aligned}$$

so, for large enough i:

$$\begin{aligned} \|b_i c_i - \alpha(\pi(b_i c_i)) &\leq \|b_i c_i - \beta(\pi(b_i))\gamma(\pi(c_i))\| \\ &\quad + \|\beta(\pi(b_i))\gamma(\pi(c_i)) - \alpha(\pi(b_i c_i))\| \\ &\leq \|b_i - \beta(\pi(b_i))\|\|c_i\| \\ &\quad + \|\beta(\pi(b_i))\|\|c_i - \gamma(\pi(c_i))\| \\ &\quad + \|\beta(\pi(b_i))\gamma(\pi(c_i)) - \alpha(\pi(b_i c_i))\| \\ &\leq \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned} \tag{2.3}$$

by a similar equation

$$\|bc - \alpha(\pi(bc))\| < \varepsilon. \tag{2.4}$$

Hence, by (2.3) and (2.4), the continuity of α and the arbitrariness of ε, it follows from Proposition 2.18, that $b_i c_i \to bc$. $\square$

Let $\mathfrak{B} = (B, G)$ be a Banach algebraic bundle. Consider a map $\lambda\colon B \to B$ and an element $g \in G$. λ is *of left order* g, if $\lambda(B_h) \subset B_{gh}$ (analogously, one defines μ of right order g).
λ is called *quasi linear*, if λ is linear on B_h for all h in g.
λ is called *bounded*, if there is some non-negative constant k in $\mathbb{R}$, such that $\|\lambda(b)\| \leq k\|b\|$ for all b in B. The smallest k with that property is called $\|\lambda\|$.

Definition 2.25. A *multiplier of* $\mathfrak{B}$ *of order* g is a pair (λ, μ), where λ and μ are continuous bounded quasi linear maps in B of left, respectively right order g, fulfilling the identities

(i) $b\lambda(c) = \mu(b)c$

(ii) $\lambda(bc) = \lambda(b)c$

(iii) $\mu(bc) = b\mu(c)$.

We call λ and μ *left-* and *right action* of $m := (\lambda, \mu)$.
We denote by $\mathcal{M}(\mathfrak{B})_g$ the collection of all multipliers of $\mathfrak{B}$ of order g. Then $\mathcal{M}(\mathfrak{B}) := \bigcup_{g\in G} \mathcal{M}(\mathfrak{B})_g$ is nearly a Banach bundle over G with a norm defined by $\|(\lambda, \nu)\|_0 = \max\{\|\lambda\|, \|\mu\|\}$ (it only lacks a suitable topology). We refer to $\mathcal{M}(\mathfrak{B})$ as the *multiplier bundle of* $\mathcal{B}$, and define the (not necessarily open) bundle projection π^0 by ${\pi^0}^{-1}(g) := \mathcal{M}(\mathfrak{B})_g$. Moreover we define $\mathcal{M}^1(\mathfrak{B}) := \{m \in \mathcal{M}(\mathfrak{B}) : \|m\| \leq 1\}$ and call it the *unit ball* of $\mathcal{M}(\mathfrak{B})$.

Remark 2.26. For any multiplier $m = (\lambda, \mu) \in \mathcal{M}(\mathfrak{B})$ and any $b \in \mathfrak{B}$ we will write $m \cdot b$ for $\lambda(b)$, and $b \cdot m$ for $\mu(b)$ respectively.

Definition 2.27. [12] A *Banach *-algebraic bundle* over a topological group G is a Banach algebraic bundle $\mathfrak{B} = (B, \pi)$ over G, together with a unary operation $*$ on B, satisfying

(i) $\pi(b^*) = (\pi(b))^{-1}$ for $b \in B$

(ii) for each $g \in G$, $*$ is conjugate linear on B_g to $B_{g^{-1}}$

(iii) $(bc)^* = c^*b^*$

(iv) $b^{**} = b$

(v) $\|b^*\| = \|b\|$

(vi) $b \mapsto b^*$ is continuous on B

Remark 2.28. Condition (i) is equivalent to the condition

(i′) $(B_g)^* \subset B_{g^{-1}}$

Remark 2.29. [12] Let Γ be as in Lemma 2.24. Then condition (vi) of the previous definition can be replaced by

(vi′) for each γ in Γ, the function $g \mapsto (\gamma(g))^*$ is continuous on G to B.

Example 2.30. [12] Let G be a topological group, define $\pi\colon \mathbb{C}\otimes G \to G$ via $\pi\colon (x,g)\mapsto g$, then $\mathfrak{B} = (\mathbb{C}\otimes G, \pi)$ with multiplication and involution defined via

$$\begin{aligned}(x,g)\cdot(x',g') &= (xx',gg')\\ (x,g)^* &= (\bar{x},g^{-1})\end{aligned}$$

is a Banach $*$-algebraic bundle, called the *group bundle of* G.

Example 2.31. [12] Fix a topological group G with neutral element e_G, a Banach $*$-algebra A and a homomorphism τ from G into the isometric $*$-automorphisms of A.
Assume that τ is strongly continuous. That is, the map $g\mapsto \tau_g(a)$ is continuous on $G\to A$ for all a in A.
By Lemma 2.1, $(a,x)\mapsto \tau_x(a)$ is continuous.
Let (B,π) be the trivial Banach bundle over G, whose constant fiber is the Banach space underlying A. That is $B = A\times G$, $\pi(a,x)=x$. Define

$$\begin{aligned}(a,x)\cdot(b,y) &= (a\tau_x(b),xy)\\ (a,x)^* &= (\tau_{x^{-1}}(a^*),x^{-1})\end{aligned}$$

then $\mathfrak{B}=(B,\pi)$ is a Banach $*$-algebraic bundle over G, called the τ*-semidirect product* of A and G.

Definition 2.32. [12] A *Fell bundle* (called C^**-algebraic bundle* by Fell himself in [12]) over a topological group G is a Banach $*$-algebraic bundle $\mathfrak{A}=(A,\pi)$ over G, such that

(i) $||b^*b|| = ||b||^2\ \forall b\in B$

(ii) $b^*b\geq 0$ in A_{e_G}, $\forall b\in B$, where e_G denotes the neutral element of G.

Remark 2.33. Note that for any Fell bundle $\mathfrak{A}=(a,\pi)$, the Fiber A_{e_G} is a C^*-algebra. This follows directly from condition (i) and (ii) of Definition 2.32 condition (i)–(vi) of Definition 2.27, condition (i)–(v) of Definition 2.21, and condition (ii)–(iv) of Definition 2.13.

Example 2.34. [12] Let G be a topological group, A be a C^*-algebra, then the τ-semidirect product of A and G is a Fell bundle.

Lemma 2.35. *[2] Every upper semi-continuous Fell bundle is a continuous Fell bundle.*

Proof. The continuity of the norm follows simply from the observation, that we can write it as a composition of continuous maps

$$a\mapsto (a,a^*)\mapsto aa^*\mapsto \|aa^*\|^{\frac{1}{2}} = \|a\|$$

where the last map is the norm of the C^*-algebra A_{e_G}, which is obviously continuous. ☐

Remark 2.36. Let $\mathfrak{B}$ be a Fell bundle, m be a multiplier of $\mathfrak{B}$ define a conjugation on $\mathcal{M}(\mathfrak{B})$ by

(i) $m^* \cdot a = (a^* \cdot m)^*$

(ii) $a \cdot m^* = (m \cdot a^*)^*$.

This turns $\mathcal{M}(\mathfrak{A})$ merely into a Fell bundle. We will refer to $\mathcal{M}(\mathfrak{A})$ as the *multiplier Fell bundle* of $\mathfrak{A}$.

2.3 An Equivalence to Fell Bundles

Remark 2.37. [9] Let G be a discrete group, then all continuity assumptions for Fell bundles can be dropped and we get the following simplified definition by Exel: a *Fell bundle* over G is a collection $\mathcal{B} = (B_g)_{g\in G}$ of closed subspaces of a C^*-algebra B, indexed by elements of G, satisfying $B_g^* = B_{g^{-1}}$, $B_g B_h \subset B_{gh}$

Let G be a discrete group with neutral element e_G, as in Definition 1.42 we can treat G as a category $\mathfrak{G}$. To define a weak group action of G in $\mathfrak{Corr}(2)$ on a C^*-algebra A, consider the functor $\natural\colon \mathfrak{G} \to \mathfrak{Corr}(2)$, defined by

$$\begin{aligned} \natural\colon \mathfrak{G}^{(0)} &\to \mathfrak{Corr}(2)^{(0)} & \qquad \natural\colon \mathfrak{G}^{(1)} &\to \mathfrak{Corr}(2)^{(1)} \\ \star &\mapsto A & g &\mapsto \alpha_g \end{aligned}$$

for any g, h in G. This yields some C^*-correspondences α_g, α_h and a C^*-correspondence isomorphism $\omega(g,h)$, such that

$$\natural(gf) \xLeftarrow{\omega(g,h)} \natural(g)\circ\natural(f) = \alpha_f \otimes \alpha_g \xRightarrow{\omega(g,f)} \alpha_{gf},$$

as well as a bigon $u\colon 1_A \Rightarrow \alpha_{e_G}$. Further

$$\natural(g)\circ\natural(g^{-1}) = \alpha_{g^{-1}} \otimes \alpha_g \xRightarrow{\omega(g,g^{-1})} \alpha_{gg^{-1}} = \alpha_{e_G} \xRightarrow{u^{-1}} 1_A \tag{2.5}$$

that is, any C^*-correspondence α_g is invertible, and so by Theorem 1.33 an $A-A$-imprimitivity bimodule. This proves the following lemma

Lemma 2.38. *A weak group action in $\mathfrak{Corr}(2)$ of a discrete group G on a C^*-algebra A is given by*

(i) $A-A$-imprimitivity modules α_g for all $g\in G$

(ii) invertible bigons $\omega(g,h)\colon \alpha_h \otimes_A \alpha_g \Rightarrow \alpha_{gh}$ for any pair of elements g,h in G.

(iii) an invertible bigon $u\colon {}_AA_A = 1_A \Rightarrow \alpha_{e_G}$,

such that α and ω satisfy (1.9) *and* (1.10).

Theorem 2.39. *[2] A group action by C^*-correspondences of a discrete group G on a C^*-algebra A is equivalent to a saturated Fell bundle $\mathfrak{A}$ over G, together with a C^*-algebra isomorphism $\varphi\colon A_{e_G} \xrightarrow{\sim} A$.*

Proof. Let $\mathfrak{A}$ be a saturated Fell bundle over a discrete group G, $\varphi\colon A_{e_G} \to A$ an isomorphism of C^*-algebras. The multiplication $\cdot\colon A_g \times A_h \to A_{gh}$ turns any fiber A_g into an $A_{e_G} - A_{e_G}$-bimodule. Moreover, by defining some inner products

$$\langle a, b\rangle_{A_{e_G}} := a^*b$$
$$_{A_{e_G}}\langle a, b\rangle := ab^*,$$

these become Hilbert $A_{e_G} - A_{e_G}$-bimodules, and since $\mathfrak{A}$ is saturated, even $A_{e_G} - A_{e_G}$-imprimitivity bimodules. Using the isomorphism φ, we can view any fiber A_g as an $A - A$-imprimitivity bimodule, hence an invertible C^*-correspondence from A to A. These will be our arrows α_g by defining $\alpha_g := A_{g^{-1}}$ for all g in G. The bigon $u\colon 1_A \Rightarrow \alpha_{e_G}$ is implemented by the C^*-algebra isomorphism

$$\varphi^{-1}\colon 1_A = A \to A_{e_G} = \alpha_{e_G},$$

which is an isomorphism of C^*-correspondences as well. Define a multiplicationmap $\mathrm{mult}\colon A_g \times A_h \to A_g \cdot A_h \subset A_{gh}$. The associativity in the Fell bundle yields

$$\mathrm{mult}(a_1 \cdot a_g \cdot a_2, a_h \cdot a_3) = a_1 \cdot m(a_g, a_2 \cdot a_h) \cdot a_3$$

for all $a_1, a_2, a_3 \in A$, $a_g \in A_g$, $a_h \in A_h$. Hence mult induces an $A - A$-bimodule isomorphism $A_g \otimes_A A_h \to A_{gh}$, which is isometric with respect to the inner products, and even unitary since $\mathfrak{A}$ is saturated. We let

$$\omega(g, h)\colon \alpha_h \otimes_A \alpha_g = A_{h^{-1}} \otimes_A A_{g^{-1}} \to A_{h^{-1}g^{-1}} = \alpha_{gh}$$

be this isomorphism. It is easy to see, that the diagrams (1.9) and (1.10) commute, hence (A, α, ω, u) defines a weak group action in $\mathfrak{Corr}(2)$.

For the converse, consider a group action (G, α, ω, u) of a discrete group G on a C^*-algebra A in $\mathfrak{Corr}(2)$. That is, $\alpha_g\colon A \to A$ are invertible C^*-correspondences (hence $A - A$-imprimitivity bimodules), and $u\colon 1_A \Rightarrow \alpha_{e_G}$ and $\omega(g, h)\colon \alpha_h \otimes_A \alpha_g \Rightarrow \alpha_{gh}$ are unitary bimodule homomorphisms.
To construct a Fell bundle, define the fibers by $A_g := \alpha_{g^{-1}}$ and use $\varphi = u^{-1}$ to identify $A \cong A_{e_G} = \alpha_{e_G}$ as an imprimitivity bimodule and, in particular, as a Banach space.
To define a multiplication on our Fell bundle, consider the unitary intertwiner $\omega(h, g)$ and define

$$\begin{aligned}\mathrm{mult}(g, h)\colon A_g \times A_h &\to A_{gh}\\ (A_g, A_h) &\mapsto \omega(h, g)(A_g \otimes_A A_h)\end{aligned}$$

$\mathrm{mult}(g, h)$ inherits bilinearity from $\omega(g, h)$ and is associative by diagramm (1.10) of Definition 1.42.
In particular, $\mathrm{mult}(e_G, e_G) =: \mathrm{mult}$ provides a multiplication on A_{e_G}, which turns A_{e_G} into an algebra. Considering diagram (1.9) of Definition 1.42, we get $\mathrm{mult}(a, b) = u^{-1}(a)b$ for all $a, b \in A_{e_G}$. Applying u^{-1} to this equation, considering that it is an $A - A$-bimodule homomorphism, we get $u^{-1}(\mathrm{mult}(a, b)) = u^{-1}(a)u^{-1}(b)$. Hence $\varphi = u^{-1}\colon A_{e_G} = \alpha_{e_G} \to A$ is an algebra homomorphism for our multiplication on A_{e_G}.

Therefore $\mathfrak{A} := (A_g)_{g\in G}$ as constructed above is a Banach algebraic bundle. To construct an involution on $\mathfrak{A}$, consider the map

$$v_g := u^{-1} \circ \omega(g^{-1}, g)\colon \alpha_g \otimes_A \alpha_{g^{-1}} \xRightarrow{\omega(g^{-1},g)} \alpha_{g^{-1}g} = \alpha_{e_G} \xRightarrow{u^{-1}} A$$

and recall the construction of $\hat{v}$ in Proposition 1.20 to see, that this yields an $A-A$-imprimitivity bimodule isomorphism $\hat{v}_g\colon A_g \to A^*_{g^{-1}}$ via

$$\begin{aligned}\hat{v}_g\colon A_g &\xrightarrow{\sim} A \otimes_A A_g \xrightarrow{\varphi_A^{-1}\otimes \mathrm{id}_{A_g}} (A^*_{g^{-1}} \otimes_A A_{g^{-1}}) \otimes_A A_g \\ &\xrightarrow{\sim} A^*_{g^{-1}} \otimes_A (A_{g^{-1}} \otimes_A A_g) \xrightarrow{\mathrm{Id}_{A^*_{g^{-1}}} \otimes_A v_g} A^*_{g^{-1}} \otimes_A A \xrightarrow{\sim} A^*_{g^{-1}}.\end{aligned}$$

Since $A^*_{g^{-1}}$ equals $A_{g^{-1}}$ as a set, we may view $\hat{v}_g$ as a map from A_g to $A_{g^{-1}}$. This defines an involution on $\mathfrak{A}$ by $a^* := \hat{v}_g(a)$. Notice that the map $A \to A^*$ is conjugate linear by construction.
To check that $(a^*)^* = a$, use Lemma 1.21 and apply $\mathcal{X} = \alpha_g$, $\mathcal{Y} = \alpha_{g^{-1}}$, $v = v_g$ and $w = v_{g^{-1}}$. The coherence laws for a weak group actions imply that the following diagram commutes:

$$\begin{tikzcd}
\alpha_g \otimes_A \alpha_{g^{-1}} \otimes_A \alpha_g \arrow[r, Rightarrow, "\omega(g^{-1},g)\otimes_A \mathrm{Id}_{\alpha_g}"] \arrow[d, Rightarrow, "\mathrm{Id}_{\alpha_g} \otimes_A \omega(g,g^{-1})"'] & \alpha_{g^{-1}g} \otimes_A \alpha_g \arrow[d, Rightarrow, "\omega(g,g^{-1}g)"'] \arrow[d, Rightarrow, bend left, "u^{-1}\otimes_A \mathrm{Id}_{\alpha_g}"] \\
\alpha_g \otimes_A \alpha_{gg^{-1}} \arrow[r, Rightarrow, "\omega(gg^{-1},g)"] \arrow[r, Rightarrow, bend right, "\mathrm{Id}_{\alpha_g} \otimes_A u^{-1}"'] & \alpha_g
\end{tikzcd}$$

Where v_g equals going clockwise, and $v_{g^{-1}}$ equals going counterclockwise. Hence, condition (1.3) of Lemma 1.21 is fulfilled, and therefore $\hat{v}_g$ and $\hat{v}_{g^{-1}}$ are inverse to each other. This yields $(a^*)^* = a$ for all a in $A_g = \alpha_{g^{-1}}$. Furthermore, let $a_g \in A_g$, $a_{g^{-1}} \in A_{g^{-1}}$ such that $a_g = \hat{v}_{g^{-1}}(a_{g^{-1}})$. Then by Remark 1.22:

$$\begin{aligned}\|a_g\|^2 = \|\langle a_g, a_g\rangle\| &= \|\langle \hat{v}_{g^{-1}}(a_{g^{-1}}), a_g\rangle\| \\ &= \|w(a_{g^{-1}} \otimes a_g)\| = \|a_{g^{-1}} \otimes a_g\| = \|a_{g^{-1}} \cdot a_g\|.\end{aligned}$$

Hence, $\|a\|^2 = \|\langle a, a\rangle\| = \|a^* \cdot a\|$. The only thing left to show that our involution anticommutes with our multiplication to see that A_{e_G} is indeed a C^*-algebra. Consider g, h in G and observe that the isomorphism

$$\alpha_{g^{-1}} \otimes_A \alpha_{h^{-1}} \otimes_A \alpha_{gh} \xRightarrow{\omega(h^{-1},g^{-1})\otimes_A \mathrm{Id}_{\alpha_{gh}}} \alpha_{(gh)^{-1}} \otimes_A \alpha_{gh} \xRightarrow{\omega(gh,(gh)^{-1})} \alpha_1 \xRightarrow{u^{-1}} A$$

induces the isomorphism $A_g \otimes A_h \to A_{(gh)^{-1}}$, that is $a_g \otimes a_h \mapsto (a_g a_h)^*$. On the other hand, we have the isomorphism

$$\begin{array}{lcl}
\alpha_{g^{-1}} \otimes_A \alpha_{h^{-1}} \otimes_A \alpha_{gh} & \xRightarrow{\mathrm{Id}_{\alpha_{g^{-1}}} \otimes_A \mathrm{Id}_{\alpha_{h^{-1}}} \otimes_A \omega(g,h)^{-1}} & \alpha_{g^{-1}} \otimes_A \alpha_{h^{-1}} \otimes \alpha_h \otimes_a \alpha_g \\
& \xRightarrow{\mathrm{Id}_{\alpha_{g^{-1}}} \otimes_A \omega(h,h^{-1}) \otimes_A \mathrm{Id}_{\alpha_g}} & \alpha_{g^{-1}} \otimes_A \alpha_1 \otimes_A \alpha_g \\
& \xRightarrow{\mathrm{Id}_{\alpha_{g^{-1}}} \otimes_A u \otimes_A \mathrm{Id}_{\alpha_g}} & \alpha_{g^{-1}} \otimes_A \alpha_g \\
& \xRightarrow{\omega(g,g^{-1})} & \alpha_{e_G} \\
& \overset{u}{\Rightarrow} & A
\end{array}$$

which induces the isomorphism $A_g \otimes_A A_h \to A_{h^{-1}g^{-1}}$ by $a_g \otimes_A a_h \mapsto a_h^* a_g^*$. Due to the coherence laws for weak group actions both isomorphisms agree, hence $(a_g a_h)^* = a_h^* a_g^*$. Thus A_{e_g} is indeed a C^*-algebra, which turns $(A_g)_{g\in G}$ into a Fell bundle. Moreover $(A_g)_{g\in G}$ is saturated, because u and ω are unitary. $\square$

2.4 Continuity of Weak Group Actions

Definition 2.40. [18] Let X be a locally compact Hausdorff space, $\{B_x\}_{x\in X}$ a family of C^*-algebras. An *upper semi-continuous C^*-bundle $\mathfrak{B}$ over X* is a triple $(X, \{B_x\}_{x\in X}, \Gamma_0(\mathfrak{B}))$, where $\Gamma_0(\mathfrak{B}) \subset \prod_{x\in X} B_x$ is a family of cross-sections, such that the following conditions are satisfied:

(i) $\Gamma_0(\mathfrak{B})$ is a C^*-algebra under pointwise operations and supremum norm

(ii) for each $x \in X$, $B_x = \{\gamma(x) : \gamma \in \Gamma_0(\mathfrak{B})\}$,

(iii) for each $\gamma \in \Gamma_0(\mathfrak{B})$ and each $\varepsilon > 0$, $\{x : \|\gamma(x)\| \geq \varepsilon\}$ is compact,

(iv) $\Gamma_0(\mathfrak{B})$ is closed under multiplication by $C_0(X)$, that is, for each $g \in C_0(X)$ and $f \in \Gamma_0(\mathfrak{B})$, the section gf defined by $gf(x) := g(x)f(x)$, is in $\Gamma_0(\mathfrak{B})$.

A *continuous C^*-bundle $\mathfrak{B}$ over x* is an upper semi-continuous C^*-bundle, such that for each $\gamma \in \Gamma_0(\mathfrak{B})$, $(x \mapsto \|\gamma(x)\|) \in C_0(X)$.

Remark 2.41. Note that condition (iii) is equivalent to the set of conditions

(i′) the map $x \mapsto \|\gamma(x)\|$ is upper-semicontinuous

(ii′) $\Gamma_0(B) \subset C_0(X, B)$

Remark 2.42. There are several ways to define a C^*-bundle. In [11], Fell defines a C^*-bundle $\mathfrak{B}$ (called C^*-algebra bundle by himself) as a Banach bundle, where each fiber B_x is a C^*-algebra. As pointed out by Nilsen in [18], this is equivalent to Definition 2.40. The necessary steps for the proof can be found in [11] and [12]. In [15], Kirchberg defines a C^*-bundle as a triple $\mathfrak{A} = (X, \pi_x\colon A \to A_x, A)$, where A is a C^*-algebra, and for each $x \in X$, A_x is a C^*-algebra and π_x is a surjective $*$-homomorphism, such that

(i) $\|a\| = \sup_{x\in X} \|a_x\|$, where $a_x = \pi_x(a)$ for each x

(ii) for $f \in C_0(X)$, $a \in A$, there is an element $fa \in A$, such that $(fa)_x = f(x)a_x$ for each $x \in X$.

It is easy to see that by identifying A with $\Gamma_0(\mathfrak{B})$ and π_x with ev_x, this definition is equivalent to Definition 2.40 by Nilsen.
We will always use the most comfortable view of an upper semi-continuous bundle.

Definition 2.43. [25] Let A be a C^*-algebra and X be a locally compact Hausdorff space. Then A is called *$C_0(X)$-algebra* if there is a $*$-homomorphism Φ_A from $C_0(X)$ into the center $\mathcal{ZM}(A)$ of the multiplier algebra $\mathcal{M}(A)$, which is nondegenerate in the sense, that

$$\Phi_A(C_0(x)) \cdot A := \mathrm{span}\{\Phi_A(f)a : f \in C_0(X), a \in A\}$$

is dense in A.

Lemma 2.44. *[18, Corollary 2.2] Let A be a $C_0(X)$-algebra. Define the ideals $I_x := \{f \in C_0(X) : f(x) = 0\}$ and $J_x := I_x A$. Then, for any a in A,*

(i) $\sup_{x\in X} \|a + J_x\| = \|a\|$

(ii) for any $\varepsilon > 0$, $\{x : \|a + J_x\| \geq \varepsilon\}$ is compact,

(iii) $x \mapsto \|a + J_x\|$ is upper semi-continuous.

Theorem 2.45. *[18] Let A be a $C_0(X)$-algebra. $I_x := \{f \in C_0(X) : f(x) = 0\}$ and $J_x := I_x A$ as in the previous lemma. Then there exists a unique upper semi-continuous C^*-bundle $\mathfrak{B}$ over X, such that*

(i) the fibers $B_x = A/J_x$

(ii) there is an isomorphism $\phi\colon A \to \Gamma_0(\mathfrak{B})$, satisfying $\phi(a)(x) = a + J_x$.

Proof. Let $B = \bigsqcup_{x\in X} A/J_x$, and let

$$S = \{f\colon X \to B : f(x) \in A/J_x \text{ and } x \mapsto \|f(x)\| \text{ is bounded}\}$$

S is a C^*-algebra under pointwise operations and supremum norm. Define $\phi : A \to S$ by the formula given in (ii), and let $\Gamma_0(\mathfrak{B}) := \operatorname{Im}\phi$.
We have to show, that the triple $(X, (A/J_x)_{x\in X}, \Gamma_0(\mathfrak{B}))$ is an upper semi-continuous bundle, hence fulfills condition (i)–(iv) of Definition 2.40.
Condition (i) is obviously satisfied, since ϕ is a C^*-homomorphism. Condition (ii) follows from the equality

$$\{\gamma(x) : \gamma \in \Gamma_0(\mathfrak{B})\} = \{a + J_x : a \in A\} = A/J_x.$$

To check condition (iii), recall that Φ_A is a non-degenerate injection, so by Lemma 2.44, for $x \in X$ and $\varepsilon > 0$, $\{x : \|a + J_x\| \geq \varepsilon\}$ is compact.
To prove condition (iv), suppose $\phi(a) \in \Gamma_0(\mathfrak{B})$, and $g \in C_0(x)$. Further, let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity in $C_0(X)$. Then, for a fixed x, $(g - g(x)e_\lambda)$ converges strictly in $\mathcal{M}(C_0(X))$. Hence $(\Phi_A(g - g(x)e_\lambda)a)$ converges in norm, and the limit is in J_x, because each $\Phi_A(g - g(x)e_\lambda)$ is in I_x. That is, $(\Phi_A(g - g(x)e_\lambda)a)$ converges to zero in A/J_x. Hence

$$\begin{aligned}\phi(\Phi_A(g)a)(x) &= \Phi_A(g)a + J_x = g(x)a + J_x \\ &= g(x)(a + J_x) = g(x)(\phi(a)(x)) = (g\phi)(a)(x).\end{aligned}$$

Thus $(X, (A/J_x)_{x\in X}, \Gamma_0(\mathfrak{B}))$ is an upper semi-continuous C^*-bundle.
Finally, ϕ is an isomorphism since it is surjective by definition, and by

$$\|\phi(a)\| = \sup_{x\in X} \|\phi(a)(x)\| = \sup_{x\in X} \|a + J_x\| = \|a\|$$

it is also injective. □

Example 2.46 ([25], [22]). Let D be a C^*-algebra, then $A := C_0(X, D)$ is a $C_0(X)$-algebra in a natural way:

$$\Phi_A(f)(a)(x) := f(x)a(x)$$

where $f \in C_0(X)$, $a \in A$.
Moreover, the associated C^*-bundle is isomorphic to the trivial bundle $(D \times X, \pi)$.

Proof. Since $A \cong C_0(X) \otimes D$, we can define a map

$$\begin{aligned}\Phi_A\colon C_0(X) &\to C_0(X) \otimes D \\ f &\mapsto f \otimes 1_{\mathcal{M}(D)}.\end{aligned}$$

Since $C_0(X)$ is commutative, $\operatorname{Im} \Phi_A \subseteq \mathcal{ZM}(C_0(X) \otimes D)$, and obviously $\operatorname{Im}(\Phi_A) \cdot C_0(X, D)$ is dense in $C_0(X, D)$. Hence, A is a $C_0(X)$-algebra.
Further, for some $x \in X$, the evaluation $\mathrm{ev}_x\colon C_0(X, D) \to D$, $f \mapsto f(x)$ is a surjective $*$-homomorphism. So there exists an isomorphism ϕ_x, such that

$$\begin{tikzcd} A \arrow[r, "\mathrm{ev}_x"] \arrow[d, "A(x)"'] & D \\ A_x \arrow[ur, dotted, "\phi_x"'] & \end{tikzcd}$$

commutes. Hence there is an isomorphism of C^*-bundles $((\phi_x)_{x \in X}, \mathrm{Id}\colon X \to X)$ between $(\bigsqcup_{x \in X} A_x, \rho\colon A_x \ni a \mapsto x)$ and $(D \times X, \pi\colon (d, x) \mapsto x)$. $\square$

Definition 2.47. Let X be a locally compact Hausdorff space, A be a C^*-algebra. A *Hilbert A-module bundle* is a Banach bundle $\mathfrak{E} = (E, \pi)$, where any fiber E_x is a Hilbert A-module, such that

(i) $R_e\colon a \mapsto e \cdot a$, is continuous on A to E for all $e \in E$

(ii) the map

$$\begin{aligned}\langle \cdot, \cdot \rangle\colon E \times_X E &\to A \\ (e, f) &\mapsto \langle e, f \rangle_{E_{\pi(e)}}\end{aligned}$$

is continuous for all $(e, f) \in E \times_X E$.

Definition 2.48. Let A be a commutative C^*-algebra, an A-module M is called *central*, if it is an $A - A$ bimodule, satisfying $a \cdot m = m \cdot a$ for all $m \in M$, $a \in A$.

Remark 2.49. Let A be a $C_0(X)$-algebra, then for any $f \in C_0(X)$, $a \in A$ we get

$$(\Phi_A(f)a)^* = a^* \Phi_A(f)^*$$

and since A is an essential ideal in $\mathcal{M}(A)$,

$$\Phi_A(f)a = a\Phi_A(f).$$

hence, A is a non-degenerate central $C_0(X)$-module with an action defined by $f \cdot a := \Phi_A(f)a$.

Definition 2.50. Let A, B be $C_0(X)$-algebras, a C^*-correspondence $\mathcal{X}$ from A to B is called *$C_0(X)$-linear*, if $\mathcal{X}$ is central as a $C_0(X)$-module.

Theorem 2.51. *Let X be a locally compact Hausdorff space, $\mathcal{E}$ be a Hilbert $C_0(X, A)$-module. Then there exists a unique upper semi-continuous Hilbert A-module bundle $\mathfrak{E}$, such that*

(i) the fibers $E_x = \mathcal{E}/J_x$, where $J_x = \mathcal{E}I_x$ with $I_x := \{f \in C_0(X, A) : f(x) = 0\}$.

(ii) there is an Hilbert $C_0(X,A)$-module isomorphism $\varphi\colon \mathcal{E} \to \Gamma_0(\mathfrak{E})$ satisfying $\varphi(e)(x) = e + \mathcal{E}/\mathcal{E}I_x$ with $e \in \mathcal{E}$, $x \in X$.

Proof. Analoguosly to Theorem 2.45 define $E := \bigsqcup_{x\in X} \mathcal{E}/\mathcal{E}I_x$, $S := \{f\colon X \to E : f(x) \in \mathcal{E}/\mathcal{E}I_x\}$, φ as in condition (ii) and $\Gamma_0(\mathfrak{E}) := \operatorname{Im}\varphi$, and show that this yields an upper semi-continuous Hilbert A-module bundle.
Since I_x is a closed ideal in $C_0(X,A)$, J_x is closed in $\mathcal{E}$, hence $(E_x)_{x\in X}$ is a family of Banach spaces. Moreover, any E_x is a $C_0(X,A)/C_0(X,A)I_x$ module. By Example 2.46, $C_0(X,A)/C_0(X,A)I_x \cong A$, with the projection $\rho_x\colon C_0(X,A) \to C_0(X,A)/C_0(X,A)I_x$ defined via $\rho_x(f) := f(x)$. Further, observe that

$$\langle e+\mathcal{E}I_x, h+\mathcal{E}I_x\rangle = \langle e,h\rangle + \langle e,\mathcal{E}I_x\rangle + \langle \mathcal{E}I_x, h\rangle + \langle \mathcal{E}I_x,\mathcal{E}I_x\rangle = \langle e,h\rangle + C_0(X,A)I_x,$$

Where the last equality comes from the fact, that $\langle\mathcal{E},\mathcal{E}\rangle I_x$ is dense in $C_0(X,A)I_x$ (cf. [16, page 5]). Thus the original inner product on $\mathcal{E}$ restricts to an inner product into $C_0(X,A)/C_0(X,A)I_x \cong A$, making any fiber E_x a Hilbert A-module.
Obviously, $\Gamma_0(\mathfrak{E})$ is a complex linear space under pointwise operations. Furthermore, let $\pi_x\colon \mathcal{E} \to \mathcal{E}/J_x$ be the canonical projection, then $\Gamma_0(\mathfrak{E})$ is a right $C_0(X,A)$-module, since

$$\phi(e\cdot f)(x) = \pi_x(e\cdot f) = \pi_x(e)\cdot\rho_x(f) = \pi_x(e)\cdot f(x) = \phi(e)(x)\cdot f(x) = (\phi(e)\cdot f)(x)$$

for all $e \in \mathcal{E}$, $f \in C_0(X,A)$.
Moreover, define a $C_0(X,A)$ valued inner product on $\Gamma_0(\mathfrak{E})$ by $\langle\phi(e),\phi(h)\rangle := x \mapsto \langle e,h\rangle + AI_x$. Observe that

$$\|\phi(e)\|^2 = \|\langle\phi(e),\phi(e)\rangle\|_\infty = \sup_{x\in X}\|\langle e,e\rangle + AI_x\| = \|\langle e,e\rangle\| = \|e\|^2.$$

Thus ϕ is an isometry, hence a Hilbert $C_0(X,A)$-module isomorphism.
Moreover, the equality

$$\begin{aligned}\Gamma_0(\mathfrak{E}) := \{\gamma(x) : \gamma\in\Gamma_0(\mathfrak{E})\} &= \{\phi(e)(x) : e\in\mathcal{E}\}\\ &= \{e+\mathcal{E}I_x : e\in\mathcal{E}\} = \mathcal{E}/\mathcal{E}I_x \end{aligned} \tag{2.6}$$

shows, that $\Gamma_0(\mathfrak{E}) = E_x$.
Note that

$$x \mapsto \|\gamma(x)\| = \|\langle\gamma(x),\gamma(x)\rangle_{E_x}\| = \|\langle\phi(e)(x),\phi(e)(x)\rangle_{E_x}\| = \|\langle e,e\rangle + AI_x\|$$

Hence, $x \mapsto \|\gamma(x)\|$ equals $x \mapsto \|\langle e,e\rangle + AI_x\|$ for some fixed e in E, so $\{x : \|\gamma(x)\| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$ by Lemma 2.44.
Now since $(E_x)_{x\in X}$ is a family of Hibert A-modules, and in particular Banach spaces, and $\Gamma_0(\mathfrak{E})$ is a set of sections, by equation 2.6, we can use Proposition 2.19 to find a topology on E, which gives us a Banach bundle $\mathfrak{E}$ with bundle space E and a space of continuous sections $\Gamma_0(\mathfrak{E})$. Note that by the compactness of $\{x : \|\gamma(x)\| \geq \varepsilon\}$, all elements of $\Gamma_0(\mathfrak{E})$ vanish at infinity. Moreover, since $\phi(e)$ is continuous in X (because $\varphi(e)\in\Gamma_0(\mathfrak{E})$) and E (since $\phi(\cdot)(x)$ is a quotient map), the maps R_e and $\langle\cdot,\cdot\rangle$ are continuous, since they are expressed in terms of ϕ and the original right action and inner product of $\mathcal{E}$. ☐

Theorem 2.52. *Let $\mathfrak{B}$ be an upper semi-continuous Hilbert A-module bundle over some locally compact Hausdorff space X. Then there is an Hilbert $C_0(X,A)$-module $\mathcal{E}$, such that $\mathfrak{B}$ is equal to the bundle $\mathfrak{E}$ as constructed in Theorem 2.51.*

Proof. Let $\mathfrak{B} = (B, \pi)$ be a Hilbert A-module bundle. Then $\Gamma_0(\mathfrak{B})$ is a Hilbert $C_0(X, A)$-module via the canonical C^*-algebra action

$$(\gamma \cdot f)(x) := \gamma(x) \cdot f(x)$$

where $\gamma \in \Gamma_0(\mathfrak{B})$, $f \in C_0(X, A)$, and $x \in X$ and the inner product given by

$$\langle \cdot, \cdot \rangle \colon (\gamma, \eta) \mapsto (x \mapsto \langle \gamma(x), \eta(x) \rangle_{B_x})$$

Now let $I_x \subset C_0(X, A)$ be the closed, two-sided ideal of functions vanishing at x, and $J_x = \Gamma_0(\mathfrak{B}) I_x$. Let further ev_x be the evaluation map from $\Gamma_0(\mathfrak{B})$ to B_x. ev_x is a projection, hence the induced map $\overline{\mathrm{ev}}_x \colon \Gamma_0(\mathfrak{B})/\ker \mathrm{ev}_x \xrightarrow{\cong} B_x$ is an isomorphism. We claim, that $\ker \mathrm{ev}_x = J_x$, which proves the assumption.
The inclusion $J_x \subset \ker(\mathrm{ev}_x)$ is clear, for the converse, assume $\gamma \in \ker(\mathrm{ev}_x)$. Let $\varepsilon > 0$ be given, then there exists a compact subset $K \subset X$, such that $\sup_{x \in X \setminus K} \|\gamma(x)\| < \varepsilon$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity in A, then there exists a $\lambda_0 \in \Lambda$, such that $\|\gamma(x) - \gamma(x) e_{\lambda_0}\| < \varepsilon$ for all $x \in K$ (cf. [16, page 5]). We choose a function $f \in C_0(X, A)$, such that $f(K) = \{e_{\lambda_0}\}$, $f(x) = 0$, and $\|f\| = 1$. Then

$$\begin{aligned}
\|\gamma - \gamma f\| &= \sup_{x \in X} \|\gamma(x) - \gamma(x) f(x)\| \\
&< \sup_{x \in X \setminus K} \|\gamma(x) - \gamma(x) f(x)\| + \varepsilon \\
&\leq \sup_{x \in X \setminus K} 2\|\gamma(x)\| + \varepsilon \\
&< 3\varepsilon
\end{aligned}$$

that is, $\gamma \in J_x$. □

Now we are able to define continuity for weak group actions by C^*-correspondences, following an idea from [2]. Given a topological group G, a weak action by C^*-correspondences is a family of C^*-correspondences $(\alpha_g)_{g \in G}$ from A to A. In particular, $(\alpha_g)_{g \in G}$ is a family of Hilbert A-modules and, since any α_g is invertible, a family of $A-A$-imprimitivity bimodules, hence in particular left Hilbert A-modules. Moreover, using the Rieffel Correspondence (Theorem 1.23) and taking into account, that obviously $\text{Ind-}\alpha_{\mathrm{g}}\, I_x = I_x$, a natural condition for continuity is to say, that the action $(\alpha_g)_{g \in G}$ is (upper semi-)continuous, if there is a $C_0(G, A) - C_0(G, A)$ imprimitivity bimodule α, such that $(G, (\alpha_g)_{g \in G}, \alpha)$ is an (upper semi-)continuous Hilbert A-module bundle, as well as an (upper semi-)continuous left Hilbert A-module bundle. That is, a bundle of $A-A$-imprimitivity bimodules.
Note that, since $C_0(G, A)$ is a $C_0(X)$-algebra, α is $C_0(X)$-linear as a C^*-correspondence.
Moreover, we need to impose some continuity conditions on the intertwiner $\omega(g, h)$. Considering Lemma 2.24 we say that $\omega(g, h) \colon \alpha_h \otimes_A \alpha_g \to \alpha_{gh}$ is continuous, if the map $(g, h) \mapsto \gamma(g)\eta(h)$ is continuous for all γ, η in α.
Now let $\mu \colon G \times G \to G$ be the multiplication map, and $\pi_1, \pi_2 \colon G \times G \to G$ the coordinate projections, the condition that $\omega(g, h)$ is continuous for all g, h in G can be rephrased into the existence of a unitary intertwiner

$$\omega \colon \pi_2^* \alpha \otimes_{C_0(G,A)} \pi_1^* \alpha \Rightarrow \mu^* \alpha$$

There are no further continuity assumptions for u. Hence we define:

Definition 2.53. [2] A continuous action of G on A by correspondences consists of a $C_0(G)$-linear correpondence α from $C_0(G, A)$ to itself, and unitary intertwiners

$$\omega\colon \pi_2^*\alpha \otimes_{C_0(G,A)} \pi_1^*\alpha \Rightarrow \mu^*\alpha$$

and $u\colon \mathrm{Id}_A \Rightarrow \alpha_{e_G}$ that satisfy analogues of (1.9) and (1.10).

The following theorem can be found in [2], even though the proof is merely sketched, and considers only one direction.

Theorem 2.54. *A continuous group action by C^*-correspondences of a locally compact group G on a C^*-algebra A is equivalent to a saturated Fell bundle $\mathfrak{A}$ over G, together with a C^*-isomorphism $\varphi\colon A_{e_G} \xrightarrow{\sim} A$.*

Proof. Let G be a topological group, $\mathfrak{A} = (A, \pi)$ be a saturated Fell bundle over G. Define $((\alpha_g)_{g\in G}, (\omega(g,h))_{g,h\in G}, u)$ as in Theorem 2.39. We have to show, that $(\alpha_g)_{g\in G}$ admits a continuous $A-A$-imprimitivity bimodule bundle. This is quite easy, since the operation $+$ is continuous on $\mathfrak{A}$, it is continuous on $(\alpha_g)_{g\in G}$. The same holds for the maps $\lambda \mapsto \lambda a$, $\lambda \in \mathbb{C}$, $a \in A$. Moreover, since $\cdot$ and $*$ are continuous on A, so are $\langle a, b\rangle_A = a^*b$, ${}_A\langle a, b\rangle = ab^*$ and R_e. Since $i\colon g \mapsto g^{-1}$ is certainly a continuous, open surjection, so is $\rho(a) := i \circ \pi(a)$, hence $((\alpha_g)_{g\in G}, \rho)$ is a continuous $A-A$-imprimitivity bimodule bundle. Finally, the continuity of $\cdot$ in $\mathfrak{A}$ yields the continuity of $(g, h) \mapsto \gamma(g)\eta(h)$ for all γ, η in $\Gamma_0((\alpha_g)_{g\in G}, \rho)$, hence the continuity of ω.

For the converse, assume that (A, α, ω, u) is a continuous weak group action by C^*-correspondences, let $\mathfrak{A}$ be the Fell bundle constructed in Theorem 2.39. $\mathfrak{A}$ is a Banach bundle by construction, and for the continuity of $\cdot$, we use again Lemma 2.24 and the continuity of ω. Moreover, the involution $*$ is continuous, if the map $g \mapsto \gamma(g)^*$ is continuous for all $\gamma \in \Gamma_0(\mathfrak{A})$. Since $\gamma(g)^* = \gamma(g^{-1})$, and the inversemap is continuous on G, this yields the continuity of $*$
Furthermore, since α is assumed to be continuous, so is $\mathfrak{A}$ $\square$

Remark 2.55. By Lemma 2.35, it suffices for α to be upper semi-continuous, to yield a continuous Fell bundle.

Chapter 3

An Extension to 2-Group

Given a continuous surjection between locally compact spaces, one can define pull backs of Banach bundles. We will introduce this construction and show that pull backs of quotient maps yield a special class of Fell bundles. Furthermore we discuss the notion of a crossed module and show the equivalence between crossed module and quotient groups. Using the fact, that crossed module generalize quotients of groups, we use our former results to define Fell bundles over crossed modules, hence 2-groups.
Given that definition, we will prove that it yields an equivalence between weak actions of discrete 2-groups in $\mathfrak{Corr}(2)$ and saturated Fell bundles over discrete 2-groups. Furthermore, we use some theory about continuity of multipliers on Fell bundles to define continuity of weak 2-group actions in $\mathfrak{Corr}(2)$, and generalize our former result.

3.1 Pull-back Fell bundles

Proposition 3.1. *[11] Let Y, Z be topological spaces and $f\colon Y \to Z$ be a surjection. f is open iff for any net $(z_i)_{i\in I}$ in Z converging to some $f(y)$, there exists a subnet $(z'_j)_{j\in J}$ for some directed set J and a net $(y_j)_{j\in J}$, such that*

(i) $f(y_j) = z'_j$ for all j in J, and

(ii) $y_j \to y$ in Y.

Proof. Assume that f is open. Let I be the directed set domain of (z_i). Define a directed set $(J, >)$ by

$$\begin{aligned} J :=& \{(i,U) : i \in I, U \subset Y : U \text{ is a neighborhood of } y\} \\ &(i,U) > (i',U') \Leftrightarrow i > i' \text{ and } U \subset U' \end{aligned}$$

Now for a $J = (i,U)$ in J, let i_j be an element of I such that $i_j > i$ and $z_{i_j} \in f(U)$; such an i_j exists, since f is open. Thus for each $j = (i,U)$ in J we can choose an element y_j in U for which $f(y_j) = z_{i_j}$. The y_j and $z'_j := z_{i_j}$ then satisfy properties (i) and (ii).
For the converse: [25] assume that f is a surjection. Suppose that U is open in Y, such that $f(U)$ is not open in Z. Then there is a net $(z_i)_{i\in I}$ such that $z_i \to f(y) \in f(U)$ with $z_i \notin f(U)$ for all $i \in I$.

By assumption, there is a subnet $(z'_j)_{j\in J}$ and a net $(y_j)_{j\in J}$ converging to y such that $f(y_j) = z'_j$
But y_j is eventually in U, hence z'_j is eventually in $f(U)$, but $z'_j = z_{N_j}$ for some $N\colon J \to I$. So this is nonsense, and thus f is open. $\square$

Definition 3.2. [12] Let G, H be topological groups, $\phi\colon H \to G$ be a continuous surjective homomorphism, and $\mathfrak{B} = (B, \pi\colon B \to G)$ be a Banach $*$-algebraic bundle over G.
Use the pullback $D := B_\pi \times_\phi H$ and the projection $\rho(b, h) := h$ to define $\phi^*\mathfrak{B} := (D, \rho\colon D \to H)$ and equip the fibers D_h with the Banach space structure under which the bijection $\iota\colon b \mapsto (b, h)$ of $B_{\phi(h)}$ onto D_h is a linear isometry. Further define a multiplication and a convolution on $\phi^*\mathfrak{B}$ via

$$
\begin{aligned}
(b, h) \cdot (b', h') &:= (bb', hh')\\
(b, h)^* &:= (b^*, h^{-1})
\end{aligned}
$$

We call $\phi^*\mathfrak{B}$ the *Banach $*$-algebraic bundle retraction of $\mathfrak{B}$ by ϕ.*

Proposition 3.3. *$\phi^*\mathfrak{B}$ is a Banach $*$-algebraic bundle*

Proof. At first, we check that ρ is an continuous open surjection. It's obvious, that ρ is indeed continuous and surjective. To check, that ρ is open, take a point (b, h) in D and a net (h_i) converging to h in H. Thus $\phi(h_i) \to \phi(h) = \pi(b)$; so by Proposition 3.1 and the openness of π we can replace (h_i) by a subnet (h_j) and find a net (b_j) in B such that $\pi(b_j) = \phi(h_j)$ for all j and $b_j \to b$. Hence $(b_j, h_j) \in D$ and $(b_j, h_j) \to (b, h)$. Since $\rho(b_j, h_j) = h_j$, another application of 3.1 shows that ρ is open. Hence $\phi^*\mathfrak{B}$ is a bundle, and even a Banach bundle, since it inherits the Banach bundle structure directly from $\mathfrak{B}$. To proof, that $\phi^*\mathfrak{B}$ is a Banach algebraic bundle, we have to check the conditions (i)–(v) of Definition 2.21.
Condition (i) is obviously fulfilled. And since $\lambda(b, h) = (\lambda b, h)$ for $\lambda \in \mathbb{C}$, $(b, h) \in D$, so is (ii).
The associativity of the multiplication follows from the associativity of the multiplications in H and $\mathfrak{B}$. Hence we get (iii).
Since ι is an isometry, we get

$$
\|(b, h)(b', h')\| = \|bb'\| \le \|b\|\|b'\| = \|(b, h)\|\|(b', h')\|
$$

and thus (iv).
The continuity of $\cdot$ follows directly from the continuity of multiplication in G and $\mathfrak{B}$, hence we have (v).
Finally we have to show, that $\phi^*\mathfrak{B}$ is a $*$-algebraic bundle, hence to check (i) – (vi) of Definition 2.27.
First of all,

$$
\rho((b, h)^*) = \rho((b^*, h^{-1})) = h^{-1} = \rho((b, h))^{-1}.
$$

Thus we got (i). Secondly,

$$
(\lambda b, h)^* = ((\lambda b)^*, h^{-1}) = (\overline{\lambda} b^*, h^{-1}) = \overline{\lambda}(b^*, h^{-1}) = \overline{\lambda}(b, h)^*
$$

which proves condition (ii). Further

$$((b,h)(b',h'))^* = (bb',hh')^* = (b'^*b^*, h'^{-1}h^{-1})$$
$$= (b'^*, h'^{-1})(b^*, h^{-1}) = (b',h')^*(b,h)^*$$

and

$$(b,h)^{**} = (b^*, h^{-1})^* = (b,h)$$

Hence (iii) and (iv) are fulfilled. At last,

$$\|(b,h)^*\| = \|b^*\| = \|b\| = \|(b,h)\|$$

and $*$ is continuous on D, since $*$ is continuous on B and the inversemap is continuous on H. Thus, $\phi^*\mathfrak{B}$ is indeed a Banach $*$-algebraic bundle. $\square$

Lemma 3.4. *Let G be a topological group, N a normal subgroup of G and $q\colon G \to G/N$ be the quotient map. Let further $\mathfrak{B} = (B,\pi)$ be a Fell bundle over G/N.*
Then the Banach $*$-algebraic retraction $q^*\mathfrak{B}$ is a Fell bundle over G.

Proof. We have to check the conditions (i) and (ii) of Definititon 2.32. Since $\mathfrak{B}$ is a Fell bundle, these follow directly from the isometric property of ι. $\square$

Theorem 3.5. *[7] Let G be a discrete group, N a normal subgroup. A Fell bundle $\mathfrak{A}$ over G is isomorphic to a pull-back bundle $q^*\mathfrak{B}$ for some Fell bundle $\mathfrak{B}$ over G/N iff there is a homomorphism $u\colon N \to \mathcal{UM}(\mathfrak{A})$ such that*

(i) $u_n \in \mathcal{M}(A)_n$ for all $n \in N$

(ii) $a_s u_n = u_{sns^{-1}} a_s$ for all $a_s \in A_s, n \in N$

Proof. Let $\mathfrak{B}$ be a Fell bundle over G/N, $q^*\mathfrak{B} = (D,\rho)$ be its pull-back bundle. Define $u_n := (1_{\mathcal{M}(B_{e_{G/N}})}, n)$. Then for $(b,h),(b',h') \in D$ we get

$$u_n \cdot (b,h) = (b,nh) \quad \text{and} \quad (b,h)\cdot u_n = (b,hn)$$

and therefore

$$((b,h)\cdot u_n) \circ (b',h') = (bb', hnh') = (b,h) \circ (u_n \cdot (b',h')).$$

Hence $u_n \in \mathcal{M}(D)_n$. Further

$$(b,h)\cdot u_n = (b,hn) = (b,hnh^{-1}h) = u_{hnh^{-1}}(b,h).$$

Thus u_n fulfills condition (i) and (ii) and u_n is indeed unitary, because $u_n u_n^* = (1_{\mathcal{M}(B_{e_{G/N}})_N}, n)(1_{\mathcal{M}(B_{e_{G/N}})_N}, n^{-1}) = (1_{\mathcal{M}(B_{e_{G/N}})_N}, e_G)$.
For the converse, assume that we have a Fell bundle $\mathfrak{A} = (A, \pi\colon A \to G)$, and a map u satisfying (i) and (ii). Let N act on A by $a \cdot n := a \cdot u_n$ and write $[a]$ for the N-orbit of $a \in A$. For $g \in G$ define

$$B_{gN} := \{[a] : a \in A_{gn} \text{ for some } n \in N\}$$

For each $g \in G$, the orbit map $a \mapsto [a]$ takes A_g bijectively onto B_{gN}, and the resulting Banach space structure on B_{gN} depends only on the coset gN. More precisely, for $a, b \in A_g$, $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}[][a+b] &= [au_n + bu_n] \\ [\lambda a] &= [\lambda a u_n]\end{aligned}$$

Thus, each fiber B_{gN} has a well defined Banach space structure, and we get a Banach bundle $\mathfrak{B}$ over G/N. Define multiplication and involution by

$$\begin{aligned}[][a][b] &= [ab] \\ [a]^* &= [a^*]\end{aligned}$$

These operations are well defined, since for $a_g \in A_g$, $a_h \in A_h$ and $n, k \in N$

$$\begin{aligned}[][(a_g u_n)(a_h u_k)] &= [a_g a_h u_{h^{-1}nhk}] = [a_g a_h] \\ [(a_g u_n)^*] &= [u_n^* a_g^*] = [a_g^* u_{g^{-1}n^{-1}g}] = [a_g^*]\end{aligned}$$

and give a Fell bundle structure on $\mathfrak{B}$: for example, if $a_g \in A_g$, $a_h \in A_h$, then $[a_g a_h] \in B_{ghN}$, hence $B_{gN} B_{hN} \subset B_{gNhN}$. To check the norm properties, observe that

$$\|[a_g]^*[a_g]\| = \|[a_g^*][a_g]\| = \|[a_g^* a_g]\| = \|a_g^* a_h g\| = \|a_g\|^2 = \|[a_g]\|^2$$

To finish the proof, we have to check, that $\phi\colon \mathfrak{A} \to q^*\mathfrak{B}$ defined by

$$\phi(a_g) := ([a_g], g) \qquad \text{for } a_g \in A_g$$

is a Fell bundle isomorphism: for example

$$\begin{aligned}\phi(a_g)\phi(a_h) &= (a_g, g)(a_h, h) = ([a_g][a_h], gh) \\ &= ([a_g a_h], gh) = \phi(a_g a_h)\end{aligned}$$

□

3.2 Crossed Modules and their actions in $\mathfrak{Corr}(2)$

Definition 3.6. Let G, H be discrete groups, a *crossed module* is some quadruple (G, H, ∂, γ), such that $\gamma\colon G \to \mathrm{Aut}(H)$ is a group homomorphism and ∂ is a equivariant homomorphism $\partial\colon H \to G$, with respect to conjugation in G, such that

(i) $\partial(\gamma_g(h)) = g\partial(h)g^{-1}$

(ii) $\gamma_{\partial(h_1)}(h_2) = h_1 h_2 h_1^{-1}$

Where we will refer to (ii) as the *Pfeiffer identity*, and call ∂ the *boundary map* of $\mathcal{C}$.

Corollary 3.7. *Let $\mathcal{C} = (G, H, \partial, \gamma)$ be a crossed module, then $\partial(H)$ is a normal subgroup of G*

Proof. The proof is a direct result of property (i). Since ∂ is a group homomorphism, $\partial(H)$ is clearly a subgroup. Further, since $g\partial(h)g^{-1} = \partial(\gamma_g(h))$, we get $g\partial(h)g^{-1} \in \partial(H)$ for all $h \in H$, $g \in G$. Hence $\partial(H)$ is normal in G. $\square$

Theorem 3.8. *A crossed module $\mathcal{C} = (G, H, \partial, \gamma)$ corresponds to a strict 2-goup $\mathfrak{G}$ via*

$$\begin{aligned}\mathfrak{G}^{(1)} &:= G\\ \mathfrak{G}^{(2)} &:= G \times H\end{aligned}$$

with horizontal and vertical multiplication defined by

$$\begin{aligned}(g_1, h_1) \cdot_h (g_2, h_2) &:= (g_1 g_2, h_1 \gamma_{g_1}(h_2))\\ (\partial(h)g, h') \circ (g, h) &:= (g, h'h)\end{aligned}$$

where (g, h) denotes the bigon $h \colon g \mapsto \partial(h)g$

Proof. We have to check the first and third identity of condition (iii) of Definition 1.34 (anything else should be clear).
For the first identity, let $(g_1, h_1), (g_2, h_2) \in \mathfrak{G}^{(2)}$, $g_1, g_2 \in \mathfrak{G}^{(1)}$, then

$$\begin{aligned}((g_1, h_1) \cdot_h (g_2, h_2))(g_1 \circ g_2) &= (g_1 g_2, h_1 \gamma_{g_1}(h_2))(g_1 g_2)\\ &= \partial(h_1 \gamma_{g_1}(h_2)) g_1 g_2\\ &= \partial(h_1) g_1 \partial(h_2) g_1^{-1} g_1 g_2\\ &= \partial(h_1) g_1 \partial(h_2) g_2\\ &= (g_1, h_1)(g_1) \circ (g_2, h_2)(g_2).\end{aligned}$$

Let $(g_1, h_1), (g_2, h_2), (\partial(h_1)g_1, h_1'), (\partial(h_2)g_2, h_2')$ be in $\mathfrak{G}^{(2)}$. Then

$$\begin{aligned}&((\partial(h_1)g_1, h_1') \cdot_h (\partial(h_2)g_2, h_2')) \circ ((g_1, h_1) \cdot_h (g_2, h_2))\\ =&(\partial(h_1)g_1\partial(h_2)g_2, h_1'\gamma_{\partial(h_1)g_1}(h_2')) \circ (g_1 g_2, h_1 \gamma_{g_1}(h_2))\\ =&(g_1 g_2, h_1' \gamma_{\partial(h_1)g_1}(h_2') h_1 \gamma_{g_1}(h_2))\\ =&(g_1 g_2, h_1' h_1 \gamma_{g_1}(h_2') h_1^{-1} h_1 \gamma_{g_1}(h_2))\\ =&(g_1 g_2, h_1' h_1 \gamma_{g_1}(h_2') \gamma_{g_1}(h_2))\\ =&(g_1 g_1, h_1' h_1 \gamma_{g_1}(h_2' h_2))\\ =&(g_1, h_1' h_1) \cdot_h (g_2, h_2' h_2)\\ =&((\partial(h_1)g_1, h_1') \circ (g_1, h_1)) \cdot_h ((\partial(h_2)g_2, h_2') \circ (g_2, h_2)).\end{aligned}$$

Conversely, let $\mathfrak{G}$ be a 2-group, denote by $H \subset \mathfrak{G}^{(2)}$ the set of bigons emanating from the unit element in $\mathfrak{G}^{(1)}$, and let $G := \mathfrak{G}^{(1)}$. Note that H is a group under horizontal multiplication. Define

$$\begin{aligned}\partial \colon H &\to G & \gamma_g \colon G &\to H\\ h &\mapsto r^{(2)}(h) & h &\mapsto ghg^{-1}.\end{aligned}$$

Then

$$\partial(\gamma_g(h)) = r^{(2)}(ghg^{-1}) = gr^{(2)}(h)g^{-1} = g\partial(h)g^{-1}$$

and

$$\gamma_{\partial(h_1)}(h_2) = r^{(2)}(h_1)h_2r^{(2)}(h_1)^{-1} = h_1h_2h_1^{-1}$$

Hence, $\mathcal{C} = (G, H, \partial, \gamma)$ is a crossed module $\square$

By Corollary 3.7, for any crossed module $\mathcal{C} = (G, H, \partial, \gamma)$, $\partial(H)$ is a normal subgroup of G, further by the previous theorem any crossed module may be seen as a 2-group. Since $(g, h)\colon g \mapsto \partial(h)g$, we get the equality $G/\partial(H) = \mathcal{G}^{(1)}/\mathcal{G}^{(2)}$. Hence a crossed module may be seen as a pull-back of the quotient group $G/\partial(H)$. To define a Fell bundle over a crossed module, and hence a 2-group we may want our Fell bundle to be isomorphic to the pull-back bundle $q^*\mathfrak{B}$ of some Fell bundle $\mathfrak{B}$ over $G/\partial(H)$, where q denotes the quotient map $q\colon G \to G/\partial(H)$. Using Theorem 3.5 this leads to the following definition:

Definition 3.9. Let $\mathcal{C} = (G, H, \partial, \gamma)$ be a crossed module over some discrete groups G, H. A *Fell Bundle* over $\mathcal{C}$ is an ordinary Fell bundle $\mathcal{A} = \{A_g\}_{g\in G}$ over G together with a group homomorphism $u\colon H \to \mathcal{UM}(\mathcal{A})$ such that

(i) $u_h \in \mathcal{UM}(\mathcal{A})_{\partial(h)}$ for all $h \in H$

(ii) $a \cdot u_h = u_{\gamma_g(h)} \cdot a$ for all $a \in A_g$, $h \in H$.

such that $s^{(2)}(b_i) = r^{(2)}(a_i)$ for $i = 1, 2$ and $a_2 \cdot_h a_1$ and $b_1 \cdot_h b_2$ are defined Let $\mathcal{C} = (G, H, \partial, \gamma)$ be a crossed module over some discrete groups G, H. By Theorem 3.8, $\mathcal{C}$ can be seen as a 2-group $\mathfrak{G}$. Then the action of $\mathcal{C}$ in $\mathfrak{Corr}(2)$ on a C^*-algebra A can be described by a functor $\natural\colon \mathfrak{G} \to \mathfrak{Corr}(2)$ via

$$\begin{array}{ccc} \natural\colon \mathfrak{G}^{(0)} \to \mathfrak{Corr}^{(0)} & \natural\colon \mathfrak{G}^{(1)} \to \mathfrak{Corr}^{(1)} & \natural\colon \mathfrak{G}^{(2)} \to \mathfrak{Corr}^{(2)} \\ \star \mapsto A & g \mapsto \alpha_g & (g,h) \mapsto \nu_{(g,h)} \end{array}$$

For the definition of the α_g, one can apply the same results as in Lemma 2.38. To describe the $\nu_{(g,h)}$, use the properties of $\natural$ again:

$$\begin{aligned} \nu_{(g,h)} \cdot_h \nu_{(g',h')} &= \natural((g,h)) \cdot_h \natural((g',h')) = \natural((g,h) \cdot_h (g',h')) \\ &= \natural(gg', h\gamma_g(h')) = \nu_{(gg', h\gamma_g(h'))} \end{aligned}$$

Further:

$$\nu_{(g,h)}(\alpha_g) = \natural((g,h)(g)) = \natural(\partial(h)g) = \alpha_{\partial(h)g}$$

and

$$\nu_{(g,h)} \circ \nu_{(\partial(h)g,h')} = \natural((g,h) \circ (\partial(h)g, h')) = \natural((g, h'h)) = \nu_{(g,h'h)}$$

Moreover,

$$\begin{aligned} \nu_{(\partial(h)g,h^{-1})} \circ \nu_{(g,h)} &= \nu_{(g,1)}\colon \alpha_g \to \alpha_g \\ \nu_{(g_1,\gamma_{g_1^{-1}}(h^{-1}))} \cdot_h \nu_{(g_2,h)} &= \nu_{(g_1g_2,e_G)}\colon \alpha_{g_1g_2} \to \alpha_{g_1g_2} \end{aligned}$$

This proves the following lemma:

Lemma 3.10. *A weak action of a crossed module $\mathcal{C} = (G, H, \partial, \gamma)$ over some discrete groups G, H is given by*

(i) $A - A$-imprimitivity modules α_g for all $g \in G$

(ii) invertible bigons $\omega(g,h)\colon \alpha_h \otimes_A \alpha_g \Rightarrow \alpha_{gh}$ *for any pair of elements* $g, h \in G$

(iii) an invertible bigon $u\colon {}_A\mathcal{A}_A = 1_A \Rightarrow \alpha_{e_G}$

(iv) invertible bigons $\nu_{(g,h)}\colon \alpha_g \Rightarrow \alpha_{\partial(h)g}$ *for all* (g,h) *in* $G \times H$

such that α *and* ω *satisfy* (1.9) *and* (1.10)*. And* ν *satisfies*

$$\nu_{(g,h)} \cdot_h \nu_{(g',h')} = \nu_{(gg', h\gamma_g(h'))}$$

and

$$\nu_{(g,h)} \circ \nu_{(\partial(h)g,h')} = \nu_{(g,h'h)}$$

for any g, g' *in* G*,* h, h' *in* H*.*

Corollary 3.11. *Let* (α, ν, ω, u) *be a weak action of a crossed module* $\mathcal{C} = (G, H, \partial, \gamma)$*. Define* $v_h := \nu_{(1,h)}$ *and* $v_h^* := v_{h^{-1}}$*, then*

(i) $v_h \in \mathcal{UM}(A)$

(ii) every $\nu_{(g,h)}$ *can be expressed in terms of* v_h *and* ω

(iii) $\alpha_{\partial(h)}(a) = v_h a v_h^*$

(iv) $\alpha_g(v_h)\alpha_1 \cong v_{\gamma_g(h)}\alpha_1$

Proof. Since v_h is an unitary equivalence of imprimitivity bimodules, we have $v_h \in \mathcal{L}(\alpha_1) = \mathcal{L}({}_A\mathcal{A}_A) = \mathcal{M}(A)$. And since v_h is unitary by definition, $v_h \in \mathcal{UM}(A)$, hence (i). Moreover, $\nu_{(g,h)} = \omega(\partial(h), g) \circ (\mathrm{Id} \otimes \mathrm{Ad}_{v_h}) \circ \omega^*(1, g)$, which proves (ii).
Further, since $v_h\alpha_1 = \alpha_{\partial(h)}$, and v_h intertwines the actions on α_g, we get $\alpha_{\partial(h)}(a) = v_h \alpha_{e_G}(a) v_h^* = v_h a v_h^*$.
Finally, we have

$$\begin{aligned}\alpha_g(v_h)\alpha_1 &= \alpha_g u_h \alpha_{g^{-1}}\alpha_1 \cong \alpha_g v_h \alpha_1 \alpha_{g^{-1}} \\ &\cong \alpha_g \alpha_{\partial(h)} \alpha_{g^{-1}} \cong \alpha_{g\partial(h)g^{-1}} = v_{\gamma_g(h)}\alpha_1.\end{aligned}$$

□

Example 3.12. Let (τ, v) be a strict group action of $\mathcal{C} = (G, H, \partial, \gamma)$ on a C^*-algebra A. That is, τ is an ordinary group action $\tau\colon G \to$ *-Aut and v is a group homomorphism $v\colon H \to \mathcal{UM}(\mathcal{A})$ satisfying

(i) $\tau_{\partial(h)}(a) = v_h a v_h^*$

(ii) $\tau_g(v_h) = v_{\gamma_g(h)}$

Now consider the τ-semidirect product bundle $\mathcal{A} = A \times_\tau G$ and define u_h by

$$\begin{aligned} u_h \cdot (a, g) &:= (a v_h^*, \partial(h) g) \\ (a, g) \cdot u_h &:= (a v_{\gamma_g(h)}^*, g\partial(h)) \end{aligned}$$

Then (A, u) becomes a Fell bundle over $\mathcal{C}$.

Proof. It is obvious that u_h by construction is a map of order $\partial(h)$. It remains to check, that u_h is indeed a unitary multiplier. This can be done by simple calculations:

$$\begin{aligned}
(a_1,g_1)((a_2,g_2)\cdot u_h) &= (a_1,g_1)\left(a_2 v_{\gamma_{g_2}(h)}, g_2\partial(h)\right)\\
&= \left(a_1\tau_{g_1}\left(a_2 v^*_{\gamma_{g_2}(h)}\right), g_1g_2\partial(h)\right)\\
&= \left(a_1\tau_{g_1}(a_2)v^*_{\gamma_{g_1g_2}(h)}, g_1g_2\partial(h)\right)\\
&= (a_1\tau_{g_1}(a_2), g_1g_2)\cdot u_h\\
&= ((a_1,g_1)(a_2,g_2))\cdot u_h
\end{aligned}$$

So $u_h \in \mathcal{M}(\mathcal{A})_{\partial(h)}$. Further we show that u_h is unitary.

$$\begin{aligned}
u_h\cdot u_h^*\cdot(a,g) &= u_h\cdot((a,g)^*\cdot u_h)^*\\
&= u_h\cdot\left(\left(\tau_{g^{-1}}(a^*), g^{-1}\right)\cdot u_h\right)^*\\
&= u_h\cdot\left(\tau_{g^{-1}}(a^*)v^*_{\gamma_{g^{-1}}(h)}, g^{-1}\partial(h)\right)^*\\
&= u_h\cdot\left(\tau_{\partial(h)^{-1}g}\left(\left(\tau_{g^{-1}}(a^*)v^*_{\gamma_{g^{-1}}(h)}\right)^*\right), \partial(h)^{-1}g\right)\\
&= u_h\cdot\left(\tau_{\partial(h)^{-1}g}\left(v_{\gamma_{g^{-1}}(h)}\tau_{g^{-1}}(a)\right), \partial(h)^{-1}g\right)\\
&= u_h\cdot\left(\tau_{\partial(h)^{-1}g}\left(v_{\gamma_{g^{-1}}(h)}\tau_{g^{-1}}(a)\right), \partial(h)^{-1}g\right)\\
&= u_h\cdot\left(\tau_{\partial(h)^{-1}g}\left(\tau_{g^{-1}}(v_h)\tau_{g^{-1}}(a)\right), \partial(h)^{-1}g\right)\\
&= u_h\cdot\left(\tau_{\partial(h)^{-1}}(v_h a), \partial(h)^{-1}g\right)\\
&= \left(\tau_{\partial(h)^{-1}}(v_h a)v_h^*, g\right)\\
&= \left(\left(v_h^*\left(v_h a\right)v_h\right)v_h *, g\right)\\
&= (a,g)
\end{aligned}$$

The calculations for $u_h^*\cdot u_h\cdot(a,g)$ are quite similar, and will be therefore omitted at this place. □

Definition 3.13. We will refer to (A,u) as constructed in 3.12 as the (τ,v)-*semidirect product* of A and $\mathcal{C}$.

Theorem 3.14. *Let $\mathcal{C}$ be a crossed module, an action of $\mathcal{C}$ on a C^*-algebra A in the category $\mathfrak{Corr}(2)$ is equivalent to a saturated Fell bundle $(\mathcal{A},u)$ together with an isomorphism of C^*-algebras $A_1 \cong A$*

Proof. Let $\mathcal{C}=(G,H,\partial,\gamma)$ act on a C^*-algebra A by (α,ν) we use the construction from Theorem 2.39 to construct a Fell bundle over the underlying 1-group. To get a Fell bundle over a 2-group, we need to construct some multipliers $u_h \in \mathcal{UM}(\mathcal{A})_{\partial(h)}$. Therefore, let $\nu_{(g,h)}\colon \alpha_g \to \alpha_{\partial(h)g}$ be the bigons induced by $(g,h)\in G\times H$ and define $u_h := (\lambda_h,\mu_h)$ by

$$\begin{aligned}
\lambda_h &:= \nu_{(g^{-1},\gamma_{g^{-1}}(h^{-1}))}\colon \mathcal{A}_g = \alpha_{g^{-1}} \to \alpha_{g^{-1}\partial(h)^{-1}} = \mathcal{A}_{\partial(h)g}\\
\mu_h &:= \nu_{(g^{-1},h^{-1})} \qquad\colon \mathcal{A}_g = \alpha_{g^{-1}} \to \alpha_{\partial(h)^{-1}g^{-1}} = \mathcal{A}_{g\partial(h)}
\end{aligned}$$

It is obvious, that λ_h, μ_h are indeed of left- respectively right order $\partial(h)$. Further, to show that $a\lambda_h(b) = \mu_h(a)b$ for $a \in \mathcal{A}_{g_1}$, $b \in \mathcal{A}_{g_2}$, calculate

$$\begin{aligned}
&a\lambda_h(b)\\
=&\omega(g_2^{-1}\partial(h)^{-1}, g_1^{-1})(a \otimes \nu_{(g_2^{-1},\gamma_{g_2^{-1}}(h^{-1}))}(b))\\
=&\omega(g_2^{-1}\partial(h)^{-1}, g_1^{-1})(\nu_{(\partial(h)^{-1}g_1^{-1},h)} \circ \nu_{(g_1^{-1},h^{-1})}(a) \otimes \nu_{(g_2^{-1},\gamma_{g_2^{-1}}(h^{-1}))}(b))\\
=&\nu_{(g_2^{-1},\gamma_{g_2^{-1}}(h^{-1}))} \cdot_h \nu_{(\partial(h)^{-1}g_1^{-1},h)}\omega(g_2^{-1}, \partial(h)^{-1}g_1^{-1})(\nu_{(g_1^{-1},h^{-1})}(a) \otimes b)\\
=&\nu_{(g_1^{-1}\partial(h)^{-1}g_2^{-1},\gamma_{g_2^{-1}}(h^{-1})g_2^{-1}(h))}\omega(g_2^{-1}, \partial(h)^{-1}g_1^{-1})(\mu_h(a) \otimes b)\\
=&\nu_{((g_1\partial(h))^{-1}g_2^{-1},\gamma_{g_2^{-1}}(e))}(\mu_h(a)b)\\
=&\mu_h(a)b.
\end{aligned}$$

Moreover, for $a \in A_{g_1}$, $b \in A_{g_2}$:

$$\begin{aligned}
\mu(ab) &= \nu_{(g_2^{-1}g_1^{-1},h^{-1})} \circ \omega(g_2^{-1}, g_1^{-1})(a \otimes_A b)\\
&= \nu_{(g_2^{-1},h^{-1})} \cdot_h \nu_{(g_1^{-1},e_G)} \circ \omega(g_2^{-1}, g_1^{-1})(a \otimes_A b)\\
&= \omega(\partial(h)^{-1}g_2^{-1}, g_1^{-1})(\nu_{(g_1^{-1},e_G)}(a) \otimes_A \nu_{(g_2^{-1},h^{-1})}(b))\\
&= a\mu(b).
\end{aligned}$$

A similar calculation shows $\lambda(ab) = \lambda(a)b$. In a final step,

$$\begin{aligned}
\lambda_{\gamma_g(h)}(a) = \nu_{(g^{-1},\gamma_g^{-1}(\gamma_g(h)^{-1}))}(a)&\\
= \nu_{(g^{-1},\gamma_g^{-1}(\gamma_g(h^{-1})))}(a) = \nu_{(g^{-1},h^{-1})}(a) = \mu_h(a)&
\end{aligned}$$

for all $a \in A_g$, and all $g \in G$. Hence $u_h \in \mathcal{UM}(A)_{\partial(h)}$ for all h in H.

Conversely, given a Fell Bundle $\mathcal{A}$ over a crossed module $\mathcal{C} = (G, H, \partial, \gamma)$, construct a group action as in Theorem 2.39. The only thing left is to define some bigons $\nu_{(g,h)}$.
To accomplish this, use the multipliers u_h to define $\nu_{(g,h)} := \mu_{h^{-1}}$, hence

$$\nu_{(g,h)} \colon \alpha_g = A_{g^{-1}} \xrightarrow{\mu_{h^{-1}}} A_{g^{-1}\partial(h)^{-1}} = \alpha_{\partial(h)g}.$$

We have to check, that this constructions produces the right horizontal and vertical multiplication, as defined in Theorem 3.8. For the vertical multiplication, that is

$$\begin{aligned}
&(\nu_{(\partial(h)g,h')} \circ \nu_{(g,h)})(\alpha_g)\\
=&(A_{g^{-1}} \cdot u_{h^{-1}}) \cdot u_{h'^{-1}}\\
=&A_{g^{-1}} \cdot u_{h^{-1}h'^{-1}}\\
=&\nu_{(g,h'h)}(\alpha_g)
\end{aligned}$$

For the horizontal multiplication consider

$$\begin{aligned}\nu_{(g_1,h_1)} \cdot_h \nu_{(g_2,h_2)}(\alpha_{g_1} \circ \alpha_{g_2}) =& \nu_{(g_1,h_2)}(\alpha_{g_1}) \circ \nu_{(g_2,h_2)}(\alpha_{g_2}) \\ =& A_{g_2^{-1}} \cdot u_{h_2^{-1}} A_{g_1^{-1}} \cdot u_{h_1^{-1}} \\ =& A_{g_2^{-1}} A_{g_1^{-1}} \cdot u_{\gamma_{g_1}(h_2^{-1})} \cdot u_{h_1^{-1}} \\ =& A_{(g_1 g_2)^{-1}} \cdot u_{(h_1 \gamma_{g_1}(h_2))^{-1}} \\ =& \nu_{(g_1 g_2, h_1 \gamma_{g_1}(h_2))}(\alpha_{g_1 g_2})\end{aligned}$$

Since u_h is unitary by definition, this yields the right bigons $\nu_{(g,h)}$. □

3.3 Continuity of weak 2-Group Actions

Let $\mathfrak{A}$ be a Fell bundle, we equip the multiplier Fell bundle $\mathcal{M}(\mathfrak{A})$ with the *strong topology.* That is, for any net $(u_\lambda)_{\lambda \in \Lambda}$ we say

$$u_\lambda \to u \Leftrightarrow u_\lambda \cdot a \to u \cdot a \text{ and } a \cdot u_\lambda \to a \cdot u \text{ in } A \text{ for all } a \in A$$

Lemma 3.15. *[10] Let G be a locally compact group, $\mathfrak{A}$ be a Fell bundle over G, then the maps $(u,a) \mapsto ua$ and $(u,a) \mapsto au$ on $\mathcal{M}^1(\mathfrak{A}) \times A$ to A are continuous.*

Proof. At first we show, that if $\gamma \in \Gamma_0(\mathfrak{A})$, then

$$(u,g) \mapsto u(\gamma(g)) \tag{3.1}$$

is continuous on $\mathcal{M}^1(\mathfrak{A}) \times G$ to $\mathfrak{A}$. Let $u_i \to u$ in $\mathcal{M}^1(\mathfrak{A})$, $g_i \to g$ in G. We may assume that g_i and g all lie in a compact subset $K \subset G$. Let $\varepsilon > 0$, $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit in $\mathfrak{A}$. We fix some index λ_0 in Λ, such that

$$\|e_{\lambda_0}\gamma(k) - \gamma(k)\| < \varepsilon \text{ for all } k \text{ in } K \tag{3.2}$$

To show that $u_i\gamma(g_i) \to u\gamma(g)$, we use Proposition 2.18. Define

$$a_i := (u_i e_{\lambda_0})\gamma(g_i) \qquad \text{and} \qquad a := (u e_{\lambda_0})\gamma(g).$$

Then $a_i \to a$, since $u_i e_{\lambda_0} \to u e_{\lambda_0}$ in A, and multiplication in A is jointly continuous. Hence our (a_i) fulfills condition (i) of Proposition 2.18. Moreover, equation (3.2) and the fact, that $\|u_i\|_0 \leq 1$ and $\|u\|_0 \leq 1$ yield condition (iii) and (iv), and of course (ii) is fulfilled by construction. Hence (3.1) is continuous. Now suppose that $u_i \to u$ in $\mathcal{M}^1(\mathfrak{A})$, and $a_i \to a$ in A. Choose $\gamma \in \Gamma_0(\mathfrak{A})$, such that $\gamma(\pi(a)) = a$. Then $u_i\gamma(\pi(a_i)) \to ua$ by the first part of the proof, and

$$\|u_i a_i - u_i\gamma(\pi(a_i))\| \leq \|a_i - \gamma(\pi(a_i))\| \to 0.$$

Hence $u_i a_i \to ua$, which proofs the assumption. The proof for the map $(u,a) \mapsto au$ works the same way. □

Corollary 3.16. *Let G be a locally compact group, $\mathfrak{A}$ be a Fell bundle over G, and $\mathcal{M}(\mathfrak{A})$ be the multiplier bundle over $\mathfrak{A}$ equipped with the strong topology. Then $\mathcal{UM}(\mathfrak{A})$ is a topological group.*

Proof. Let $u \in \mathcal{M}(\mathfrak{A})$, and $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of $\mathfrak{A}$. Then for each $a \in A$, we have $(ue_\lambda)a = u(e_\lambda a) \to ua$ and $a(ue_\lambda) = (au)e_\lambda \to au$, hence $ue_\lambda \to u$ strongly in $\mathcal{M}(\mathfrak{A})$. That is, A is strongly dense in $\mathcal{M}(\mathfrak{A})$. So there is a continuous extension of the maps in Lemma 3.15 to $\mathcal{M}^1(\mathfrak{A}) \times \mathcal{M}(\mathfrak{A})$, which restricted to $\mathcal{UM}(\mathfrak{A}) \times \mathcal{UM}(\mathfrak{A})$ yields a continuous multiplication on $\mathcal{UM}(\mathfrak{A})$. Moreover, for some net $u_i \to u$ in $\mathcal{UM}(\mathfrak{A})$, we have $u_i^* a = (a^* u_i)^* \to (a^* u)^* = u^* a$ for all a in A. Doing the same calculations for au^*, we see that the involution on $\mathcal{M}(\mathfrak{A})$ is strongly continuous, and since $u^* = u^{-1}$ for unitaries, so is the inversion on $\mathcal{UM}(\mathfrak{A})$. Thus $\mathcal{UM}(\mathfrak{A})$ is a topological group with respect to the strong topology. □

Corollary 3.17. *$\mathcal{UM}(\mathfrak{A})$ is strongly closed in $\mathcal{M}^1(\mathfrak{A})$.*

Proof. Let $u_i \to u$ in $\mathcal{M}^1(\mathfrak{A})$, such that $u_i \in \mathcal{UM}(\mathfrak{A})$ for all i, then by the continuity of the involution and multiplication on $\mathcal{M}^1(\mathfrak{A})$, we have $1 = u_i u_i^* \to uu^*$, and hence $u^* = u^{-1}$. □

Definition 3.18. Let G be a locally compact group, $\mathfrak{A}$ be a Fell bundle over G, we define the *uniform-on-compacta* topology on $\Gamma(\mathfrak{A})$ by saying, that a net γ_i converges to γ, if and only if for each compact subset K in G, $\sup_{g\in K} \|\gamma_i(g) - \gamma(g)\| \to 0$

Now let G be a locally compact group, $\mathfrak{A}$ be a Fell bundle over G, and $K \subset G$ be compact, then we define a semi-norm p_K on $\Gamma(\mathfrak{B})$ by

$$p_K(\gamma) := \sup_{g\in K} \|\gamma(g)\|$$

The collection of all these seminorms generates the uniform-on-compacta topology for $\Gamma(\mathfrak{B})$.
Moreover, we want to define an action of $\mathcal{M}(B)$ on $\Gamma(\mathfrak{B})$. Let $u \in \mathcal{M}(\mathfrak{A})_g$, and $\gamma \in \Gamma(\mathfrak{A})$, define:

$$\begin{aligned}(u\gamma)(h) &:= u(\gamma(g^{-1}h)\\ (\gamma u)(h) &:= (\gamma(hg^{-1}))u\end{aligned}$$

Clearly $u\gamma$ and γu are in $\Gamma_0(\mathfrak{A})$, and $(uv)\gamma = u(v\gamma)$ and $\gamma(uv) = (\gamma u)v$ for $v \in \mathcal{M}(\mathfrak{B})$. Further it is easy to verify, that

$$\begin{aligned}p_K(u\gamma) &\le \|u\|_0 p_{g^{-1}K}(\gamma)\\ p_K(\gamma u) &\le \|u\|_0 p_{Kg^{-1}}(\gamma)\end{aligned}$$

Proposition 3.19. *[10] Let (u_i) be a net in $\mathcal{M}^1(\mathfrak{A})$, u an element of $\mathcal{M}^1(\mathfrak{A})$. It is a necessary and sufficient for $u_i \to u$ strongly that the following two conditions hold:*

(i) Let $\pi^0(u_i) \to \pi^0(u)$ in G.

(ii) $u_i\gamma \to u\gamma$ and $\gamma u_i \to \gamma u$ uniformly-on-compacta for all γ in $\Gamma_0(\mathfrak{B})$.

Proof. Assume that $u_i \to u$ strongly, that is $u_i a \to ua$ and $au_i \to au$ for all a in A. Since $(u_i a)$ can be seen as a net in A converging to ua, it is necessary that $\pi(u_i a) \to \pi(ua)$, which proves the necessity of (i). Let $\gamma \in \Gamma_0(\mathfrak{B})$, let $K \subset G$

be compact, and let $\varepsilon > 0$. By Lemma 3.15, the map $(v, g) \mapsto v\gamma(\pi^0(v)^{-1}g)$ is continuous on $\mathcal{M}^1(\mathfrak{A}) \times G$ to B. So

$$(v, g) \mapsto \|v\gamma(\pi^0(v)^{-1}g) - u\gamma(\pi^0(u)^{-1}g)\|$$

is continuous on $\mathcal{M}^1(\mathfrak{A}) \times G$ and vanishes for $v = u$. Consequently, by the compactness of K, there is a strong neighborhood U of u in $\mathcal{M}^1(\mathfrak{A})$, such that $\|v\gamma(\pi^0(v)^{-1}g) - u\gamma(\pi^0(u)^{-1}g)\| \leq \varepsilon$ for all v in U and g in K. But this says $p_K(v\gamma - u\gamma) < \varepsilon$ for all v in U. Since K and ε are arbitrary, this implies $u_i\gamma \to u\gamma$ uniformly on compacta. Similarly $\gamma u_i \to \gamma u$, so (ii) holds.
Conversely, assume (i) and (ii), and let a be an arbitrary element of A. Choose $\gamma \in \Gamma_0(\mathfrak{A})$, so that $\gamma(\pi(a)) = a$. Let $g_i := \pi(u_i a)$, $g = \pi(ua)$. By (i), $g_i \to g$. Since $u_i\gamma \to u\gamma$ uniformly on compacta and $g_i \to g$, we have $u_i\gamma(g_i) \to u\gamma(g)$. But $(u_i\gamma)(g) = u_i\gamma(\pi^0(u_i)^{-1}g_i) = u_i\gamma(\pi(a)) = u_i a$, and $(u\gamma)(g) = u\gamma(\pi^0(u)^{-1}g) = u\gamma(\pi(a)) = ua$. So $u_i a \to ua$ in A. Similarly $au_i \to au$. So $u_i \to u$ strongly. □

Proposition 3.20. *[10] Let g be in G, and (u_i) be a net of elements of $\mathcal{UM}(\mathfrak{B})$, such that $\pi^0(u_i) \to g$. Suppose further that there exist functions M and L on $\Gamma(\mathfrak{A})$ to $\Gamma(\mathfrak{A})$ such that $u_i\gamma \to L(\gamma)$ and $\gamma u_i \to M(\gamma)$. Then there exists a multiplier u in $\mathcal{UM}(\mathfrak{A})$, such that $\pi^0(u) = g$ and $u\gamma = L(\gamma)$ and $\gamma u = M(f)$ for all $\gamma \in \Gamma_0(\mathfrak{A})$.*

Proof. Let $h \in G$, $\eta \in \Gamma_0(\mathfrak{A})$ and $\eta(h) = 0_h$, then

$$\|L(\eta)(gh)\| = \lim_i \|(u_i\eta)(\pi^0(u_i)h)\| = \lim_i \|u_i\eta(h)\| \leq \|\eta(h)\| = 0.$$

Hence, $L(\eta)(gh)$ depends only on $\eta(h)$. Note further, that L and M are necessarily linear, therefore we can define a quasi-linear map $\lambda\colon A \to A$ of left order g as follows:

$$\lambda(\eta(h)) := L(\eta)(gh) \tag{3.3}$$

Note that by the preceding argument,

$$\lambda(a) = \lim_i u_i a \tag{3.4}$$

for all a in A. In particular

$$\|\lambda(a)\| \leq \|a\| \tag{3.5}$$

To show that λ is continuous, suppose $a_i \to a = \gamma(h)$ in A, where $\gamma \in \Gamma_0(\mathfrak{A})$. Then, if $h_i = \pi(a_i)$, we have by (3.5)

$$\|\lambda(a_i) - \lambda(\gamma(h_i))\| \leq \|a_i - \gamma(h_i)\| \to 0$$

Also, $h_i \to h$, so

$$\lambda(\gamma(h_i)) = L(\gamma(gh_i)) \to L(f)(gh) = \lambda(a).$$

Hence λ is continuous. Similarly, we define a continuous bounded quasi linear function $\mu\colon B \to B$ of right order g, such that

$$\mu(\gamma(h)) = M(f)(hg) \tag{3.6}$$

$$\mu(a) = \lim_i au_i \tag{3.7}$$

$$\|\mu(a)\| \leq \|a\| \tag{3.8}$$

Finally, if $a, b \in A$, we have by (3.4) and (3.7)

$$\mu(a)b = \lim_i a(u_i b) = a\lambda(b)$$

Thus, $u := (\lambda, \mu) \in \mathcal{M}(\mathfrak{A})$. By (3.5) and (3.8) $u \in \mathcal{M}^1(\mathfrak{A})$. Moreover, by (3.3) and (3.6) we have $u\gamma = L(\gamma)$ and $\gamma u = M(\gamma)$ for all $\gamma \in \Gamma(\mathfrak{A})$, and by Corollary 3.17 u is unitary. $\square$

Definition 3.21. Let $\mathcal{C} = (G, H, \partial, \gamma)$ be a crossed module, we say $\mathcal{C}$ is a *topological* crossed module, if G and H are topological groups, ∂ is a continuous group homomorphism and γ is strongly continuous.

Definition 3.22. Let $\mathcal{C} = (G, H, \partial, \gamma)$ be a topological crossed module. A Fell bundle over $\mathcal{C}$ is *continuous*, if the underlying Fell bundle over G is continuous, and if the map $u\colon H \to \mathcal{UM}(\mathfrak{A})$ is a continuous group homomorphism with respect to the strong topology on $\mathcal{UM}(\mathfrak{A})$.

Now we are ready to define continuity for a weak action $(A, \alpha, \omega, u, \nu)$ of a topological crossed module on a C^*-algebra in $\mathfrak{Corr}(2)$. Considering the previous discussion, we say that $\nu_{(g,h)}$ is continuous, if the map $(h, g) \mapsto \nu_{(g,h)}(\eta(\partial(h)^{-1}g))$ is continuous for all $\eta \in \alpha$. On the one hand, this expresses the close relation of ν to multipliers, on the other hand it ensures that $\nu_{(\pi(a_i),h_i)}(a_i) \to \nu_{(\pi(a),h)}(a)$ for $h_i \to h$ in H, and $a_i \to a$ in A. Hence we define:

Definition 3.23. Let $(A, \alpha, \omega, u, \nu)$ be a weak group action of a topological crossed module $\mathcal{C} = (G, H, \partial, \gamma)$ in the category $\mathfrak{Corr}(2)$. We say $(A, \alpha, \omega, u, \nu)$ acts *continuously* on A, if (A, α, ω, u) is a continuous action by G, and if the map

$$(h, g) \mapsto \nu_{(g,h)}(\eta(\partial(h)^{-1}g)) := (\nu_{(g,h)}\eta)(h)$$

is continuous from $H \times G$ to α_g for all $\eta \in \alpha$.

Theorem 3.24. *Let $\mathcal{C}$ be a topological crossed module, a continuous action of $\mathcal{C}$ on a C^*-algebra A in the category $\mathfrak{Corr}(2)$ is equivalent to a continuous saturated Fell bundle $(\mathcal{A}, u)$ together with an isomorphism of C^*-algebras $A_1 \cong A$*

Proof. Let $(A, \alpha, \omega, u, \nu)$ be a weak action of a crossed module $\mathcal{C} = (G, H, \partial, \gamma)$, apply 3.14 to construct a Fell bundle $\mathfrak{A}$ over $\mathcal{C}$, moreover use Theorem 2.54 to show, that the underlying Fell bundle over G is continuous. All we have to check is the continuity of the map $h \mapsto u_h$

Recall that for $u_h = (\lambda_h, \mu_h)$ we have $\mu_h := \nu_{(g^{-1},h^{-1})}$. Moreover, for any section $\eta \in \Gamma_0(\mathfrak{B})$, let $\eta^i := \eta \circ i$ be the corresponding cross-section in α, where i denotes the inversion on G. Now by Theorem 3.20 $\mu_{h_i} \to \mu$ if there exists a continuous function $M\colon \Gamma_0(\mathfrak{B}) \to \Gamma_0(\mathfrak{B})$ such that $\mu_{h_i}(\eta) \to M(\eta)$. Then $\mu(\eta) = M(\eta)$. Let $h_i \to h$ in H, using the continuity of $\nu_{(g,h)}$, we have

$$\begin{aligned}
\mu_{h_i}(\eta)(g) &= \mu_{h_i}(\eta(g\partial(h_i)^{-1})) \\
&= \nu_{(g^{-1},h_i^{-1})}(\eta^i(\partial(h_i)g^{-1}) \\
&\to \nu_{(g^{-1},h^{-1})}(\eta^i(\partial(h)g^{-1}) \\
&= \mu_h(\eta(g\partial(h)^{-1})) \\
&= \mu_h(\eta)(g)
\end{aligned}$$

Thus we have $M(\eta) = \mu_h(\eta)$, and therefore

$$\mu(\eta(g)) = M(\eta)(g\partial(h)) = \mu_h(\eta(g\partial(h)\partial(h)^{-1}) = \mu_h(\eta(g)),$$

hence $\mu = \mu_h$ and therefore $\mu_{h_i} \to \mu_h$. The proof for λ_h works similarly.

Conversely, let $\mathcal{C} = (G, H, \partial, \gamma)$ be a topological crossed module, $\mathfrak{A} = (A, \pi)$ a saturated continuous Fell bundle over $\mathcal{C}$. Define a crossed module action $(A, \alpha, \omega, u, \nu)$ as in Theorem 3.14. By Theorem 2.54, α and ω are continuous. Recall that $\nu_{(g,h)} = \mu_{h^{-1}}$. Since $h \mapsto u_h$ is continuous, and $\mu_{h_i}(a_i) \to \mu_h(a)$ by Lemma 3.15, we have

$$\begin{aligned}
\nu_{(g_i,h_i)}(\gamma^i)(g_i) &= (\mu_{h_i^{-1}}\gamma)(g_i^{-1}) \\
&= \mu_{h_i^{-1}}(\gamma(g_i^{-1}\partial(h_i)^{-1})) \\
&\to \mu_{h^{-1}}(\gamma(g^{-1}\partial(h)^{-1})) \\
&= \nu_{(g,h)}(\gamma^i(\partial(h)^{-1}g)) \\
&= \nu_{(g,h)}(\gamma^i)(g)
\end{aligned}$$

which proves the assumption. □

Chapter 4

An Extension to Groupoids

We are aiming on to extend our results in Section 2 to groupoids. Hence, we are starting with a brief discussion of locally compact, and especially *r-discrete* groupoids. Then we define actions of r-discrete groupoids in $\mathfrak{Corr}(2)$ and Fell bundles over groupoids and generalize the results of Section 2 to the case of r-discrete groupoids.

4.1 Properties of r-discrete Groupoids

Definition 4.1. [21] A *topological groupoid* consists of a groupoid $\mathcal{G}$ and a topology $\mathcal{T}$ compatible with groupoid structure, that is

(i) the inversion $x \mapsto x^{-1}$ is continuous and

(ii) the multiplication $(g, h) \mapsto gh$ is continuous with respect to the product topology on $\mathcal{G}^{(1)} \times \mathcal{G}^{(1)}$.

Remark 4.2. [21] For any topological groupoid, the following statements hold:

(i) s, r are continuous

(ii) the inverse map i is a homeomorphism

(iii) $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}^{(1)}$ via the embedding $\iota\colon x \mapsto 1_x$.

(iv) if $\mathcal{G}^{(0)}$ is Hausdorff, $\mathcal{G}^{(1)}{}_s\times_r \mathcal{G}^{(1)}$ is closed in $\mathcal{G}^{(1)} \times \mathcal{G}^{(1)}$

Definition 4.3. [21] Let $\mathcal{G}$ be a locally compact groupoid. A *left Haar system* for $\mathcal{G}$ consists of measures $\{\lambda^x, x \in \mathcal{G}^{(0)}\}$ on $\mathcal{G}$, such that

(i) $\operatorname{supp}\lambda^x = \mathcal{G}^x$

(ii) (continuity) for any $f \in C_c(\mathcal{G})\}$, the map $x \mapsto \int f \mathrm{d}\lambda^x$ is continuous and has compact support

(iii) (left invariance) for any $x \in \mathcal{G}^{(1)}$, $f \in C_C(\mathcal{G})$ holds $\int f(xy)\mathrm{d}\lambda^{s(x)}(y) = \int f(y)\mathrm{d}\lambda^{r(x)}(y)$

Definition 4.4. [19] A *locally compact groupoid* is a topological groupoid $\mathcal{G}$ satisfying

(i) $\mathcal{G}^{(0)}$ is locally compact Hausdorff in the relative topology inherited from $\mathcal{G}$

(ii) there is a countable family $\mathcal{A}$ of compact Hausdorff subsets of $\mathcal{G}$, such that the family $\{\mathring{A} : A \in \mathcal{A}\}$ of interiors of members of $\mathcal{A}$ form a basis of the topology of $\mathcal{G}$

(iii) every $\mathcal{G}^x$ is locally compact Hausdorff in the relative topology inherited from $\mathcal{G}$

(iv) $\mathcal{G}$ inherits a left Haar system $\{\lambda^x\}$

Remark 4.5. [19] Definition 4.4 yields the following statements:

- by (i), every singleton subset of $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}^{(0)}$. Thus by continuity of r and s, each $\mathcal{G}^x$, $\mathcal{G}_x$ is closed in $\mathcal{G}$.
- by (iii), every singleton subset of $\mathcal{G}$ is closed, and the topology of $\mathcal{G}$ has a countable basis. Further $\mathcal{G}^{(0)}$, $\mathcal{G}^x$, and $\mathcal{G}_x$ have a countable bases.

Proposition 4.6. *[19] Let $\mathcal{G}$ be a locally compact groupoid. Then r and s are open maps from $\mathcal{G}$ to $\mathcal{G}^{(0)}$.*

Proof. Let U be an open subset of $\mathcal{G}$. By Definition 4.4, U is a union of open Hausdorff subsets of $\mathcal{G}$, an thus Hausdorff too. Let $g \in U$, $f \in C_C(U)$ with $f \geq 0$ and $f(g) > 0$. Define f^0 on $\mathcal{G}^{(0)}$ by $f^0(x) := \int f \mathrm{d}\lambda^x$. By Definition 4.3 $f^0 \in C_C(\mathcal{G}^{(0)})$ and $f^0(r(g)) > 0$. Since f^0 vanishes outside $r(U)$, the open subset $\{x \in \mathcal{G}^{(0)} : f(x) > 0\} = f^{-1}((0, \infty))$ is contained in $r(U)$. So $r(U)$ is a union of open subset and therefore open in $\mathcal{G}^{(0)}$. Since $s(U) = r(U^{-1})$, $s(U)$ is also open in $\mathcal{G}^{(0)}$. □

Definition 4.7. [19] Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{G}^{op}$ be the family of open Hausdorff subsets $U \subset \mathcal{G}$, such that the restrictions $r_{|_U}$, $s_{|_U}$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. $\mathcal{G}$ is called *r-discrete*, if $\mathcal{G}^{op}$ is a basis for the topology of $\mathcal{G}$.

Remark 4.8. There are several equivalent definitions of r-discrete groupoids. Renault defines a locally compact groupoid to be r-discrete, if $\mathcal{G}^{(0)}$ is open in $\mathcal{G}^{(1)}$ (cf. [21]). It is easy to see, that this is a direct result of our definition, since $\mathcal{G}^{(0)} = \bigcup_{A \in \mathcal{G}^{op}} r(A)$ is open in $\mathcal{G}$ by Proposition 4.6. The converse is shown in by Renault in [21, Proposition 2.8].

Lemma 4.9. *[21] Let $\mathcal{G}$ be an r-discrete groupoid*

(i) For any $x \in \mathcal{G}^{(0)}$, $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete spaces.

(ii) If a Haar system exists, it is essentially the counting measure

Proof. (i) [19]Let $g \in \mathcal{G}^x$, then there is an $U \subset \mathcal{G}^{op}$, such that $g \in U$. Since r is injective on U, the singleton $\{g\} = \mathcal{G}^x \bigcap U$ is open in $\mathcal{G}^x$

(ii) [21] Let $\{\lambda^x\}$ be a left Haar system. Since $\mathcal{G}^x$ is discrete and λ^x has support $\mathcal{G}^x$, every point in $\mathcal{G}^x$ has positive λ^x-measure. Let $f = \lambda(\chi_{\mathcal{G}^{(0)}})$, where $\chi_{\mathcal{G}^{(0)}}$ is the characteristic function of $\mathcal{G}^{(0)}$. It is continuous and positive. Replacing λ^x by $f(x)^{-1}\lambda^x$, we may assume that $\lambda^x(\{g\}) = 1$ for any x. Then by invariance, $\lambda^y(\{h\}) = 1$ for any $h \in \mathcal{G}_g^h$.

□

Considering the previous lemma, we may regard an r-discrete groupoids as the groupoid analogue to discrete groups. Especially any countable discrete group G, viewed as a groupoid $\mathcal{G}$ with an one elemental object space is r-discrete.

4.2 Weak Groupoid Actions in $\mathfrak{Corr}(2)$

Let $\mathcal{G}$ be an r-discrete groupoid, $\mathfrak{A} = (A, \pi\colon A \to \mathcal{G}^{(0)})$ be a C^*-bundle in the category $\mathfrak{Corr}(2)$. We can define an action of $\mathcal{G}$ on $\mathfrak{A}$ as usual as a homomorphism of 2-categories $\natural\colon \mathcal{G} \to \mathfrak{Corr}(2)$, such that

$$\begin{aligned} \natural\colon \mathcal{G}^{(0)} &\to \mathfrak{Corr}(2)^{(0)} & \qquad \natural\colon \mathcal{G}^{(1)} &\to \mathfrak{Corr}(2)^{(1)} \\ x &\mapsto A_x & g &\mapsto \alpha_g \end{aligned}$$

where A_x denotes the fiber over x and α_g is an $A_{s(g)} - A_{r(g)}$-imprimitivity bimodule.
Further for any $(g,f) \in \mathcal{G}^{(1)}{}_{s(g)}\times_{r(f)}\mathcal{G}^{(1)}$, there exists an $A_{s(f)} - A_{r(g)}$-imprimitivity bimodule isomorphism $\omega(g,f)$, such that

$$\natural(gf) \xLeftarrow{\omega(g,f)} \natural(g) \circ \natural(f) = \alpha_f \otimes \alpha_g \xRightarrow{\omega(g,f)} \alpha_{gf}$$

and bigons $u_x\colon 1_{A_x} \Rightarrow \alpha_{1_x}$ for all $x \in \mathcal{G}^{(0)}$, such that for $y \in \mathcal{G}^{(0)}$, and all $g \in \mathcal{G}_x^y$ we get

$$\natural(1_x) = \natural(gg^{-1}) \xLeftarrow{\omega(g,g^{-1})} \alpha_{g^{-1}} \otimes \alpha_g \xRightarrow{\omega(g,g^{-1})} \alpha_{1_x} \xRightarrow{u_x} 1_{A_x}$$

Moreover, our $\omega(g,f)$ and 1_x have to satisfy some coherence laws similar to (1.9) and (1.10). The only difference is that we have to pay some extra attention to composability. This proves the following lemma:

Lemma 4.10. *A weak groupoid action in $\mathfrak{Corr}(2)$ of an r-discrete groupoid $\mathcal{G}$ on a C^*-algebra A is given by*

- *(i) $A_{s(g)} - A_{r(g)}$-imprimitivity bimodules α_g for all $g \in \mathcal{G}^{(1)}$,*
- *(ii) invertible bigons $\omega(g,h)\colon \alpha_h \otimes_{A_{r(h)}} \alpha_g \Rightarrow \alpha_{gh}$ for all (g,h) in $\mathcal{G}^{(1)}{}_{s(g)}\times_{r(h)} \mathcal{G}^{(1)}$, and*
- *(iii) invertible bigons $u_x\colon 1_{A_x} \Rightarrow \alpha_{1_x}$,*

such that any $\omega(g,h)$, u_x fulfills some analogues to the coherence laws (1.9) *and* (1.10)*, with some extra attention to composability.*

Definition 4.11. [17] Let $\mathcal{G}$ be a second countable, locally compact Hausdorff groupoid. A *Fell bundle* over $\mathcal{G}$ is a second countable Banach bundle $\mathfrak{B} = (B, \pi\colon B \to \mathcal{G})$, such that

- (i) For any $x \in \mathcal{G}^{(0)}$, B_x is a C^*-algebra and for every $g \in \mathcal{G}^{(1)}$, B_g is an invertible correspondence from $B_{r(g)}$ to $B_{s(g)}$.
- (ii) For every $(g,h) \in \mathcal{G}^{(2)}$, where $\mathcal{G}^{(2)}$ denotes the set of pairs of composable morphisms in $\mathcal{G}^{(1)}$, exists an unitary equivalence U_h^g between $B_g \otimes_{B_{s(g)}} B_h$ and B_{gh} which induces an associative, bilinear product

$$\begin{aligned} m\colon \mathfrak{B}^{(2)} &\to \mathfrak{B} \\ (b_1, b_2) &\mapsto U_{\pi(b_2)}^{\pi(b_1)}(b_1 \otimes b_2) \end{aligned}$$

Where $\mathfrak{B}^{(2)} := \{(b_1, b_2) \in B \times B : (\pi(b_1), \pi(b_2)) \in \mathcal{G}^{(2)}\}$.

(iii) There exists an involution on B, such that

a) $\pi(b^*) = \pi(b)^{-1}$

b) $b \mapsto b^*$ is conjugate linear

c) $(m(b_1, b_2))^* = m(b_2^*, b_1^*)$

d) for $b_1, b_2 \in B_g$ holds

$$\begin{aligned} m(b_1^*, b_2) &= \langle b_1, b_2 \rangle_{B_{s(g)}} \\ m(b_1, b_2^*) &= {}_{B_{r(g)}}\langle b_1, b_2 \rangle \end{aligned}$$

where we tacitly use the isomorphism $B_{1_x} \cong B_x$

(iv) The mappings m and the involution are continuous

A Fell bundle is called *saturated*, if B_g is full as a $B_{r(g)}$ Hilbert module as well as a $B_{s(g)}$ Hilbert module.

Example 4.12. [17] Let $\mathcal{G}$ be a groupoid, $\mathfrak{A} = (A, \pi : A \to \mathcal{G}^{(0)})$ a C^*-bundle over $G^{(0)}$. Assume that $\mathcal{G}$ acts on $\mathfrak{A}$ by a homomorphism $\alpha : \mathcal{G} \to \mathrm{Aut}(\mathfrak{A})$. Assume that α is continuous in the sense, that $\tilde{\alpha} : s^*\mathfrak{A} \to r^*\mathfrak{A}$ is continuous, where

$$\begin{aligned} s^*\mathfrak{A} &:= \{(a, g) : a \in A_{s(g)}\} \\ r^*\mathfrak{A} &:= \{(a, g) : a \in A_{r(g)}\} \end{aligned}$$

are the pull-back bundles and where $\tilde{\alpha}(a, g) = (\alpha_g(a), g)$
When we set $\mathfrak{B} := s^*\mathfrak{A}$ with $\rho(a, g) = g$, we obtain a Fell bundle over $\mathcal{G}$, where the product and involution are given by

$$\begin{aligned} (a_1, g_1) \cdot (a_2, g_2) &:= (\alpha_{g_2^{-1}}(a_1)a_2, g_1 g_2) \\ (a, g)^* &:= (\alpha_g(a^*), g^{-1}) \end{aligned}$$

The obtained Fell bundle $\mathfrak{B}$ is called the *crossed product of* $\mathfrak{A}$ *by* $\mathcal{G}$ and denoted by $\mathfrak{B} = \mathfrak{A} \rtimes_\alpha \mathcal{G}$.

Theorem 4.13. *Let $\mathcal{G}$ be an r-discrete groupoid. A weak groupoid action of $\mathcal{G}$ by correspondences on a C^*-bundle $\mathfrak{A} = (A, \pi : A \to \mathcal{G}^{(0)})$ is equivalent to a saturated Fell bundle $\mathfrak{B} = (B, \rho : B \to \mathcal{G})$ over $\mathcal{G}$ with isomorphisms $\varphi_x : B_{1_x} \xrightarrow{\sim} A_x$ for all $x \in \mathcal{G}^{(0)}$.*

Proof. Starting with a groupoid action $(\mathfrak{A}, \alpha, \omega, u)$, we have to generalize the proof of Theorem 2.39 and check condition (i) to (iv) of Definition 4.11.
First of all, for any $g \in \mathcal{G}^{(1)}$, we identify $B_g \in \mathfrak{B}$ with $\alpha_{g^{-1}}$. Since α_g is an $A_{s(g)} - A_{r(g)}$-imprimitivity bimodule for all $g \in \mathcal{G}^{(1)}$, B_g is an $A_{r(g)} - A_{s(g)}$-imprimitivity bimodule for all $g \in \mathcal{G}^{(1)}$. Further, for any $x \in \mathcal{G}^{(0)}$, identify $B_x \in \mathfrak{B}$ with A_x and implement the isomorphisms φ_x, using the identity bigons u_x, by $\varphi_x = u_x^{-1} : B_{1_x} = \alpha_{1_x} \to 1_{A_x} = A_x$ for all x in $\mathcal{G}^{(0)}$. That way $\mathfrak{B}$ covers all properties from Theorem 4.11, condition (i).
For (ii), the $\omega(g, f)$ yield a unitary equivalences U_g^f (cf. [16], Theorem 3.5: any

surjective, A-linear isometry between two Hilbert-modules $\mathcal{E}_A$, $\mathcal{F}_A$ is an unitary equivalence) by

$$B_g \otimes B_f = \alpha_{g^{-1}} \otimes \alpha_{f^{-1}} \xrightarrow{\omega(f^{-1},g^{-1})} \alpha_{f^{-1}g^{-1}} = B_{gf},$$

where the associativity of the multiplication $m\colon (b_1, b_2) \to U^{\pi(b_1)}_{\pi(b_2)}$ is given by the coherencelaw (1.10).
To check condition a)–d) of (iii), recall the definition of $\hat{v}_g$ and define again $*\colon x \mapsto \hat{v}_g(x)$. Since $\hat{v}_g\colon B_g \to B_{g^{-1}}$, the identity a) is obvious. b) follows directly from the definition of the dual and c) has been already shown by proving Theorem 2.39.
In order to show d), we use Remark 1.22. For $b_1, b_2 \in B_g$, $v_g = u^{-1}_{r(g)} \circ \omega(g^{-1}, g)$ and $w_g = v_{g^{-1}} = u^{-1}_{s(g)} \circ \omega(g, g^{-1})$, we get

$$\begin{aligned} m(b_1^*, b_2) = \quad & \omega(g, g^{-1})(b_1^* \otimes b_2) & = u_{s(g)} \circ w_g(b_1^* \otimes b_2) \\ = \quad & u_{s(g)}(\langle \hat{v}_{g^{-1}}(b_1^*), b_2\rangle_{s(g)}) & = u_{r(g)}(\langle b_1, b_2\rangle_{s(g)}). \end{aligned}$$

The second equation follows from c) and property (iii) of Definition 1.1.
To check (iv), recall from Lemma 4.9 that for any $x \in \mathcal{G}^{(0)}$, $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete. Now $\mathcal{G}^{(1)}{}_s\times_r \mathcal{G}^{(1)} = \bigsqcup_{x\in\mathcal{G}^{(0)}} \mathcal{G}_x \times \mathcal{G}^x$, thus $\mathcal{G}^{(1)}{}_s\times_r \mathcal{G}^{(1)}$ is r-discrete over $\mathcal{G}^{(0)}$, and therefore m is continuous. Moreover, since the inversion i is a homeomorphism, the continuity of the involution can be shown in terms of cross-sections as in Theorem 3.24.

Now let $\mathfrak{B} = (B, \rho\colon B \to \mathcal{G})$ be a groupoid Fell bundle. Since for any $x \in \mathcal{G}^{(0)}$, B_x is a C^*-algebra, this yields a C^*-bundle

$$\mathfrak{A} = (A := B \cap \pi^{-1}(\mathcal{G}^{(0)}), \pi_{|_A}\colon A \to \mathcal{G}^{(0)}).$$

Define $\alpha_g := B_{g^{-1}}$, this is by definition a $B_{s(g)} - B_{r(g)}$-imprimitivity bimodule and hence an $A_{s(g)} - A_{r(g)}$-imprimitivity bimodule. Let 1_x be the identity in $\mathcal{G}^x_x$, this yields a $B_x - B_x$-imprimitivity bimodule B_{1_x} which is isomorphic to ${}_{B_x}\mathcal{B}_{xB_x}$. We can use φ_x to define a unit bigon

$$u_x\colon 1_{A_x} = A_x \xrightarrow{\varphi_x^{-1}} B_{1_x} = \alpha_{1_x}$$

For the bigons $\omega(g, h)$, let $(g, h) \in \mathcal{G}_s\times_r \mathcal{G}$ and define

$$\omega(g, h)\colon \alpha_h \otimes \alpha_g = B_{h^{-1}} \otimes B_{g^{-1}} \xrightarrow{U^{h^{-1}}_{g^{-1}}} B_{h^{-1}g^{-1}} = \alpha_{gh}$$

By definition of m and U, this makes $\omega(g, h)$ a bilinear, associative, unitary equivalence, and thus a bigon in $\mathfrak{Corr}(2)$. $\square$

Chapter 5

An Extension to 2-Groupoids

Analogously to Chapter 3, we start with the construction of a quotient groupoid, and show that pull backs of Fell bundles by a quotient map yield a special class of a Fell bundle. Moreover, we define a crossed module of r-discrete groupoids and show that these yield 2-groupoids. Then we combine our results of all previous sections to show, that weak actions of r-discrete 2-groupoids in $\mathfrak{Corr}(2)$ are equivalent to saturated Fell bundles.

5.1 Quotientgroupoids and their Pull-Back Bundles

Definition 5.1. [1] Let $\mathcal{G}$ be a groupoid, $\mathcal{G}$ is called *totally disconnected*, if $\mathcal{G}_x^y = \emptyset$ for all x, y in $\mathcal{G}^{(0)}$, such that $x \neq y$.
$\mathcal{G}$ is called *discrete*, if $\mathcal{G}$ is totally disconnected, and $\mathcal{G}_x^x = \{1_x\}$ for all x in $\mathcal{G}^{(0)}$
A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is called *wide*, if $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$.

Remark 5.2. Please note, that in some papers a totally disconnected groupoid is meant to be a topological groupoid with (topologically) totally disconnected objectspace. This is not the case in this thesis.

Definition 5.3. [1] Let $\mathcal{G}$, $\mathcal{H}$ be groupoids, $\theta\colon \mathcal{G} \to \mathcal{H}$ a groupoidmap. Define $\ker(\theta)$ to be the wide subgroupoid of $\mathcal{G}$, whose morphism are

$$\ker(\theta)^{(1)} := \{g \in \mathcal{G}^{(1)} : \theta(g) = 1_x \text{ for some } x \in \mathcal{H}^{(0)}\}.$$

We call $\ker(\theta)$ the *kernel* of $\mathcal{G}$.
Further, for any subgroupoid $\mathcal{X}$ of $\mathcal{G}$, we say θ *annihilates* $\mathcal{X}$, if $\theta(\mathcal{X})$ is a discrete subgroupoid of $\mathcal{H}$.

Definition 5.4 ([14]). Let $\mathcal{G}$ be a groupoid. Two elements $x, y \in \mathcal{G}^{(0)}$ are called *connected*, if $\mathcal{G}_x^y \neq \emptyset$. A subgroupoid $\mathcal{H}$ is called *connected*, if all elements in $\mathcal{H}^{(0)}$ are connected. And a subgroupoid $\mathcal{H}$ is called *component* of $\mathcal{G}$, if it is a maximal connected subgroupoid.

Definition 5.5 ([14]). Let $\mathcal{G}$ be a groupoid, a subgroupoid $\mathcal{H}$ is called *normal*, iff

(i) $\mathcal{H}$ contains all identity elements of $\mathcal{G}$

(ii) $h \in \mathcal{H}_x^x$, $g \in \mathcal{G}_x^y$ implies $ghg^{-1} \in \mathcal{H}_y^y$.

Let $\mathcal{G}$ be a groupoid, $\mathcal{H}$ be a normal subgroupoid of $\mathcal{G}$. The components of $\mathcal{H}$ define a partition on $\mathcal{G}^{(0)}$. We write $[x]$ for the equivalence class containing $x \in \mathcal{G}^{(0)}$ and $\mathcal{G}/\mathcal{H}^{(0)}$ for the set of equivalence classes.
$\mathcal{H}$ also defines an equivalence relation on $\mathcal{G}^{(1)}$ as follows: $g \equiv g'(\operatorname{mod} \mathcal{H})$ iff $g = h_1 g' h_2$ for some $h_1, h_2 \in \mathcal{H}^{(1)}$ (we denote the set of equivalence classes by $\mathcal{G}^{(1)}/\mathcal{H}$. Two equivalent arrows of $\mathcal{G}$ must have there sources in the same component of $\mathcal{H}$, and similarly for their targets. So to each class $[g]$ of arrows can be assigned a unique source and target in $\mathcal{G}/\mathcal{H}^{(0)}$. We define a multiplication on $\mathcal{G}/\mathcal{H}^{(1)}$ as follows: let $[g], [f] \in \mathcal{G}/\mathcal{H}^{(1)}$, the product $[g][f]$ is defined iff there exists $g_1 \in [g]$, $f_1 \in [f]$, such that $g_1 f_1$ is defined in $\mathcal{G}$. Then $[g][f] := [g_1 f_1]$.

Proposition 5.6. *[14] This multiplication is associative and well defined*

Proof. Let $g_1, f_1, [g], [f]$ be as above and $g_2 \in [g]$, $f_2 \in [f]$ such that $g_2 f_2$ is defined in $\mathcal{G}$. Then

$$g_2 = h_{g1} g_1 h_{g2} \qquad \text{and} \qquad f_2 = h_{f1} f_1 h_{f2}$$

, where $h_{g1}, h_{g2}, h_{f1}, h_{f2} \in \mathcal{H}$, and

$$g_2 f_2 = h_{g1} g_1 h_{g2} h_{f1} f_1 h_{f2}$$

in $\mathcal{G}$. Since $g_1 f_1$ is defined in $\mathcal{G}$, $h_{g2} h_{f1}$ lies in a isotropygroup of $\mathcal{H}$, so

$$v = f_1^{-1} h_{g2} h_{f1} f_1$$

is defined and lies in $\mathcal{H}$. Hence

$$g_2 f_2 = h_{g1} g_1 f_1 v h_{f_2} \equiv g_1 f_1 (\operatorname{mod} \mathcal{H}).$$

Now for $[g]\colon [y] \mapsto [z]$, $[f]\colon [w] \mapsto [x]$ in $\mathcal{G}/\mathcal{H}$, the product $[g][f]$ is defined iff there is an $h \in \mathcal{H}$, such that ghf is defined in $\mathcal{G}$, that is, iff $[x] = [y]$, and then we have $[g][f] = [ghf]\colon [w] \mapsto [z]$. Moreover, if $([e][f])[g]$ is defined, then $eh_1 fh_2 g$ is defined for some $h_1, h_2 \in \mathcal{H}$ □

Hence, $\mathcal{G}/\mathcal{H}$ is a category whose identities are the components of $\mathcal{H}$. Moreover, we can define a map $\pi\colon \mathcal{G} \to \mathcal{G}/\mathcal{H}$, $g \mapsto [g]$, $x \mapsto [x]$ for all x in $\mathcal{G}^{(0)}$, g in $\mathcal{G}^{(1)}$. This map is surjective, hence $\mathcal{G}/\mathcal{H}$ is a groupoid, since $\mathcal{G}$ is a groupoid.

Definition 5.7. [14] We call $\mathcal{G}/\mathcal{H}$ the *quotient* of $\mathcal{G}$ over $\mathcal{H}$, and π the associated *quotient map.*

Lemma 5.8. *Let $\mathcal{G}$ be a groupoid, $\mathcal{N}$ be a normal, totally disconnected subgroupoid of $\mathcal{G}$. Then for all g in $\mathcal{G}^{(1)}$ holds $[g] = \mathcal{N}_{r(g)}^{r(g)} g$.*

Proof. By definition, $g' \in [g]$ iff $g = n_1 g' n_2$ for some $n_1 \in \mathcal{N}_{r(g')}^{r(g)}$, $n_2 \in \mathcal{N}_{s(g)}^{s(g')}$. Since $\mathcal{N}$ is totally disconnected, we have $r(g') = r(g)$ and $s(g') = s(g)$. Further, the normality of $\mathcal{N}$ implies $\mathcal{N}_{r(g)}^{r(g)} g = g \mathcal{N}_{s(g)}^{s(g)}$. So for $n_2' \in \mathcal{N}_{r(g)}^{r(g)}$, we get

$$\begin{aligned} & g = n_1 g' n_2 \\ \Leftrightarrow \quad & g n_2^{-1} = n_1 g' \\ \Leftrightarrow \quad & n_2' g = n_1 g' \\ \Leftrightarrow \quad & n_1^{-1} n_2' g = g' \\ \Leftrightarrow \quad & g' \in \mathcal{N}_{r(g)}^{r(g)} g. \end{aligned}$$

$\square$

Definition 5.9. Let $\mathfrak{B} = (B, \pi)$ be a Fell bundle over an r-discrete groupoid $\mathcal{G}$, and $g \in \mathcal{G}^{(1)}$. A *multiplier of order* g is a pair of bimodule morphisms $m = (\lambda, \mu)$ fulfilling

(i) $\lambda(B_f) = B_{gf}$ for all $f \in \mathcal{G}^{s(g)}$ and $\mu(B_f) = B_{fg}$ for all $f \in \mathcal{G}_{r(g)}$.

(ii) $b_1 \otimes_{r(g)} \lambda(b_2) \cong \mu(b_1) \otimes_{s(g)} b_2$ for all $b_1 \in B_{f_1}$, $b_2 \in B_{f_2}$ with $f_1 \in \mathcal{G}_{r(g)}$ and $f_2 \in \mathcal{G}^{s(g)}$.

(iii) $\lambda(b_1 \otimes b_2) \cong \lambda(b_1) \otimes b_2$ and $\mu(b_1 \otimes b_2) \cong b_1 \otimes \mu(b_2)$.

We denote the set of multipliers of order g by $\mathcal{M}(\mathfrak{B})_g$ and the *set of multipliers on* $\mathfrak{B}$ by $\mathcal{M}(\mathfrak{B})$.

Remark 5.10. Let $\mathfrak{B} = (B, \pi)$ be a Fell bundle, $m = (\lambda, \mu) \in \mathcal{M}(\mathfrak{B})_g$, $g \in \mathcal{G}^{(1)}$. Note that the further theorem implies

(i) $\lambda(B_{s(g)}) = B_g$ and $\mu(B_{r(g)}) = B_g$.

(ii) $b_1 \cdot \lambda(b_2) = \mu(b_1) \cdot b_2$ with $b_1 \in B_{r(g)}$, $b_2 \in B_{s(g)}$, and $\cdot$ denoting the action of $B_{s(g)}$ respectively $B_{r(g)}$ on the imprimitivity bimodule B_g.

(iii) $\lambda(b_1 b_2) = \lambda(b_1) \cdot b_2$ and $\mu(b_1' b_2') = b_1' \cdot \mu(b_2')$ for $b_1, b_2 \in B_{s(g)}$ and $b_1', b_2' \in B_{r(g)}$,

by using the equivalence $B_x = B_{1_x}$ for all x in $\mathcal{G}^{(0)}$.

Remark 5.11. Let $1_x \in \mathcal{G}$. Then $\mathcal{M}(\mathfrak{B})_{1_x} \cong \mathcal{M}(B_x)$, where the latter denotes the usual multiplier C^*-algebra of B_x. Further for $g \in \mathcal{G}_x^x$, $\mathcal{M}(B_{1_x})_g = \mathcal{M}(B_g)$, where the latter denotes the set of multipliers on the imprimitivity bimodule A_g. Finally, for $\mathcal{G}^{(0)} = \{\star\}$, let $\mathfrak{A}$ be the Fell bundle over the group $\mathcal{G}^{(1)}$ such that $\mathfrak{A} \cong \mathfrak{B}$, then $\mathcal{M}(\mathfrak{B}) \cong \mathcal{M}(\mathfrak{A})$, where the latter denotes the multiplier bundle on $\mathfrak{A}$.
Hence our definition generalizes any former definition of multipliers.

Remark 5.12. As usual, for any multiplier $m = (\lambda, \mu)$, we write $m \cdot a$ to denote $\lambda(a)$, and $a \cdot m$ for $\mu(a)$.
Further we define the usual conjugation by $m^* \cdot a := (a^* \cdot m)^*$ and $a \cdot m^* := (m \cdot a^*)^*$.

Theorem 5.13. *Let $\mathcal{G}$ be an r-discrete groupoid, $\mathcal{N}$ be a normal, totally disconnected subgroupoid of $\mathcal{G}$. A Fell bundle $\mathfrak{A}$ over $\mathcal{G}$ is isomorphic to a pull back bundle $q^*\mathfrak{B}$ for some Fell bundle $\mathfrak{B}$ over $\mathcal{G}/\mathcal{N}$, iff there is a continuous homomorphism $u\colon \mathcal{N}^{(1)} \to \mathcal{UM}(\mathfrak{A})$, such that*

(i) $u_n \in \mathcal{M}(A)_n$ for all $n \in \mathcal{N}^{(1)}$

(ii) $a_g u_n = u_{gng^{-1}} a_g$ for all $a_g \in A_g$, $n \in \mathcal{N}^{(1)}$

Proof. Let $\mathfrak{B}$ be a Fell bundle over $\mathcal{G}/\mathcal{N}$, $q^*\mathfrak{B}$ the corresponding pull-back bundle. Now let $n \in \mathcal{N}_x^x$. As in Theorem 3.5 we can define $u_n := (1_{\mathcal{M}(\mathfrak{B})_{1_x}}, n)$ to be our unitary multipliers. Since $\mathcal{N}_x^x$ is a group, we can do the same calculations as before, with some extra attention to composability, to show that

$u_n \in \mathcal{UM}(q^*\mathfrak{B})$.

Conversely, by Lemma 5.8, $[g] = \mathcal{N}_x^x g$ for all $g \in \mathcal{G}^x$. Hence the construction of Theorem 3.5 works generally identical in our new setting. □

Proposition 5.14. *[1] Let $\mathcal{G}$ be a groupoid, $\mathcal{N}$ be a normal, totally disconnected subgroupoid of $\mathcal{G}$ and $q\colon \mathcal{G} \to \mathcal{G}/\mathcal{N}$ the quotient map. Then q annihilates $\mathcal{N}$.*

5.2 Crossed Modules over Groupoids

Definition 5.15. [3] A continuous action γ of a topological groupoid $\mathcal{G}$ on a bundle $\mathfrak{H} = (h, \pi)$ of topological groups is a map $\gamma\colon \mathcal{G} \to \mathrm{Aut}(h)$ such that γ_g for $g \in \mathcal{G}_x^y$ is a group isomorphism from H_x to H_y.

Definition 5.16. [3] A *crossed module over a topological groupoid $\mathcal{G}$ and a group bundle $\mathfrak{H}$* is a quadruple $(\mathcal{G}, \mathfrak{H}, \partial, \gamma)$, where $\mathfrak{H} = (h_x)_{x \in \mathcal{G}^{(0)}}$ is a bundle of topological groups over the object space $\mathcal{G}^{(0)}$, $\partial\colon \mathfrak{H} \to \mathcal{G}$ is a continuous homomorphism, and $\gamma\colon \mathcal{G} \to \mathrm{Aut}(\mathfrak{H})$ is a continuous action, such that

$$\partial(\gamma_g(h)) = g\partial(h)g^{-1} \qquad \forall (g, h) \in G_s \times_\pi H \tag{5.1}$$

$$\gamma_{\partial(h)}(k) = hkh^{-1} \qquad \forall (h, k) \in H \times_{\mathcal{G}^{(0)}} H \tag{5.2}$$

Remark 5.17. Please note, that condition (i) of the former definition implies, that the boundary map ∂ has to act as the identity on the objectspace $\mathcal{G}^{(0)}$. Hence the image $\partial(\mathcal{H})$ is a subgroupoid of $\mathcal{G}$, having the same identities as $\mathcal{G}$.

Theorem 5.18. *A crossed module $\mathcal{C} = (\mathcal{G}, \mathfrak{H}, \partial, \gamma)$ yields a strict-2-groupoid $\mathcal{G}_\mathcal{C}$ via*

$$\mathcal{G}_\mathcal{C}^{(0)} = \mathcal{G}^{(0)}$$
$$\mathcal{G}_\mathcal{C}^{(1)} = \mathcal{G}^{(1)}$$
$$\mathcal{G}_\mathcal{C}^{(2)} = \mathcal{G}^{(1)}{}_r\times_\pi H$$

with a horizontal multiplication defined by

$$(g_1, h_1) \cdot_h (g_2, h_2) = (g_1 g_2, h_1 \gamma_{g_1}(h_2))$$

where $(g_1, g_2) \in \mathcal{G}^{(1)}{}_s \times_r \mathcal{G}^{(1)}$, $h_1 \in H_{r(g_1)}$, and $h_2 \in H_{r(g_2)}$ and a vertical multiplication defined by

$$(\partial(h)g, h') \circ (g, h) := (g, h'h)$$

where $g \in \mathcal{G}^{(1)}$, $h, h' \in H_r(g)$, and (g, h) denotes the bigon $g \mapsto \partial(h)g$

Proof. The calculations to check the coherence laws are identical to the ones done in the proof of Theorem 3.8, so we skip those here. What is left to check is, that the multiplications are actually defined.
For the horizontal multiplication, $(g_1, g_2) \in \mathcal{G}^{(1)}{}_s \times_r \mathcal{G}^{(1)}$, hence $g_1 g_2$ is defined and an element of $\mathcal{G}_{s(g_2)}^{r(g_1)}$. Further, since $h_2 \in H_{r(g_2)} = H_{s(g_1)}$, and $\gamma_{g_1} : H_{s(g_1)} \xrightarrow{\sim} H_{r(g_1)}$, $\gamma_{g_1}(h_2) \in H_{r(g_1)}$. Since $h_1 \in H_{r(g_1)}$, $h_1 \gamma_{g_1}(h_2)$ is defined

and in $H_{r(g_1)}$. Hence $(g_1g_2, h_1\gamma_{g_1}h_2) \in \mathcal{G}_r \times_\pi H = \mathcal{G}_\mathcal{C}^{(1)}$. Thus the horizontal multiplication is defined and closed.
For the vertical multiplication, note that from condition (1) of Definition 5.16 one can deduce, that for $h \in H_x$, $\partial(h) \in \mathcal{G}_x^x$. Hence $\partial(h)g \in \mathcal{G}_{s(g)}^{r(g)}$. And since $(\partial hg, h') \in \mathcal{G}_r \times_\pi H$, $h' \in H_{r(g)}$. Thus $h'h$ is defined and an element of $H_{r(g)}$. Further we get $(g,h), (\partial(h)g, h'), (g, h'h) \in \mathcal{G}_{s(g)}^{r(g)} \times H_{r(g)}$. So $\circ$ makes perfect sense as a vertical multiplication. $\square$

Corollary 5.19. *By the former Theorem, we can define a weak action of a crossed module $\mathcal{C}$ over a groupoid $\mathcal{G}$ and a bundle $\mathfrak{H}$ on a C^*-bundle $\mathfrak{A}$ over $\mathcal{G}^{(0)}$ in the $\mathfrak{Corr}(2)$-category as a (weak) 2-morphism $\natural\colon \mathcal{G}_\mathcal{C} \to \mathfrak{Corr}(2)$, such that*

$$\natural\colon \mathcal{G}_\mathcal{C}^{(0)} \to \mathfrak{Corr}(2)^{(0)} \qquad \natural\colon \mathcal{G}_\mathcal{C}^{(1)} \to \mathfrak{Corr}(2)^{(1)} \qquad \natural\colon \mathcal{G}_\mathcal{C}^{(2)} \to \mathfrak{Corr}(2)^{(2)}$$
$$x \mapsto A_x \qquad\qquad g \mapsto \alpha_g \qquad\qquad (g,h) \mapsto \nu_{(g,h)}.$$

Where α_g are invertible C^*-correspondences from $A_{s(g)}$ to $A_{r(g)}$ and $\nu_{(g,h)}$ are C^*-correspondence isomorphisms, such that $\nu_{(g,h)}\colon \alpha_g \to \alpha_{\partial(h)g}$, which inherit the vertical and horizontal multiplication from $\mathcal{G}_\mathcal{C}$.
Futher, the functor yields some invertible bigons $\omega(g_1,g_2)\colon \alpha_{g_1}\alpha_{g_2} = \alpha_{g_2} \otimes_{s(g_2)} \alpha_{g_1} \Rightarrow \alpha_{g_1g_2}$ and $u_x\colon 1_{A_x} = {}_{A_x}\mathcal{A}_{xA_x} \Rightarrow \alpha_1$, which satisfy some analogues to 1.9 and 1.10.

5.3 Weak actions of 2-Groupoids in $\mathfrak{Corr}(2)$

Let $\mathcal{C} = (\mathcal{G}, \mathfrak{H}, \partial, \gamma)$ be a crossed module, and assume $\mathcal{G}$ to be r-discrete and the fibers H_x to be discrete groups. Analogously to the group case, condition (i) of Definition 5.16 implies that $\partial(\mathfrak{H})$ is a normal subgroupoid of $\mathcal{G}$. Condition (i) of Definition 5.5 is fulfilled by Remark 5.17. To check condition (ii), let $h \in H_x$ (and hence $\partial(h) \in \partial(\mathfrak{H})_x^x$), $g \in \mathcal{G}_x^y$. Now since $\gamma_g\colon H_x \xrightarrow{\sim} H_y$, we get

$$g\partial(h)g^{-1} = \partial(\gamma_g(h)) \in \partial(H_y) \subset \partial(\mathfrak{H})_y$$

Hence, the quotient $\mathcal{G}/\partial(\mathfrak{H})$ is a groupoid.
Since $\mathfrak{H}$ is a totally disconnected groupoid, so is $\partial(\mathfrak{H})$. Hence the quotient map $q\colon \mathcal{G} \to \mathcal{G}/\partial(\mathfrak{H})$ acts as identity on $\mathcal{G}^{(0)}$. Further, by Lemma 5.8, the 2-morphisms of the 2-groupoid description of $\mathcal{C}$ are consistent with the equivalence classes of $\mathcal{G}/\partial(\mathfrak{H})$. Hence, equivalently to the group case, the crossed module $\mathcal{C}$ models the quotient $\mathcal{G}/\partial(\mathfrak{H})$.
Using Theorem 5.13, this leads to the following definition of a Fell bundle over $\mathcal{C}$:

Definition 5.20. Let $\mathcal{C} = (\mathcal{G}, \mathfrak{H}, \partial, \gamma)$ be a crossed module over an r-discrete groupoid $\mathcal{G}$ and a group bundle $\mathfrak{H} = (H, \pi\colon H \to \mathcal{G}^{(0)})$ with discrete fibers.
A Fell bundle over $\mathcal{C}$ is an ordinary Fell bundle $\mathfrak{B}$ over $\mathcal{G}$ together with homomorphisms $u\colon H \to \mathcal{UM}(\mathfrak{B})$, such that

(i) $u_h \in \mathcal{UM}(\mathfrak{B})_{\partial(h)}$ for all $h \in H$

(ii) $b \cdot u_h \cong u_{\gamma_g(h)} \cdot a$ for all $b \in B_g$, $h \in H$.

Theorem 5.21. *A weak action by C^*-correspondences of a crossed module $\mathcal{C}$ over some r-discrete groupoid $\mathcal{G}$ and a discrete group bundle $\mathfrak{H}$ on a C^*-algebra bundle $\mathfrak{A}$ is equivalent to a Fell bundle $\mathfrak{B}$ over $\mathcal{C}$, together with an isomorphism $\varphi\colon B_{1_x} \to A_x$.*

Proof. The construction should generalize Theorem 3.14 as well as Theorem 4.13. As a starting point, we use the construction of Theorem 4.13 and proof the equivalence of the corresponding 2-morphisms and unitary multipliers as in Theorem 3.14.
Starting with a Fell Bundle $\mathfrak{B}$ over some crossed module $\mathcal{C} = (\mathcal{G}, \mathfrak{H}, \partial, \gamma)$, we apply the construction of Theorem 4.13 such that for all $g, f \in \mathcal{G}$ we get some arrows $\alpha_g := B_{g^{-1}}$ and some bigons $\omega(g,h)\colon \alpha_g\alpha_f \Rightarrow \alpha_{gf}$. And for all $x \in \mathcal{G}^{(0)}$, we get some bigons $u_x\colon 1_{A_x} \Rightarrow \alpha_1$.
Analogously to the construction of Theorem 3.14, we define our bigons $\nu_{(g,h)}$ via

$$\begin{aligned} \nu_{(g,h)}\colon \alpha_g &\to \alpha_{\partial(h)g} \\ a &\mapsto a \cdot u_h \end{aligned}$$

The same calculations as before lead to the result, that these define indeed some bigons satisfying the necessary conditions.
Conversely, given a weak action of a crossed module $\mathcal{C}$, applying the construction of Theorem 4.13 to construct a Fell bundle over the groupoid $\mathcal{G}$ and using the definition made in the proof of Theorem 3.14 to get some unitary multipliers will lead to the right results. $\square$

Chapter 6

Outlook

There are several possibilities on how to continue this work. Most obviously, one might try to topologize the results of Chapter 4 and 5. Since an (upper semi-)continuous Hilbert A-module bundle is equivalent to a Hilbert $C_0(X, A)$-Module, where $C_0(X, A)$ is exactly the cross-section algebra over the trivial bundle $A \times X$. One might assume, that an (upper semi-)continuous Hilbert Module bundle $\mathfrak{E}$, consisting of a family of Hilbert A_x-modules and a C^* bundle $\mathfrak{A}$, such that any E_x is an Hilbert A_x module is equivalent to a Hilbert module over the cross-sectional algebra over $\mathfrak{A}$. Provided this construction works, one may define continuity of groupoid actions analogously to continuity of group actions.

More algebraically, one might try to extend the work to actions of 2-categories or inverse semi-groups. That is, weaken the invertibility of arrows and bigons or to generalize the construction of the cross-sectional algebra to Fell bundles over 2-groups and 2-groupoids and analyze its properties.

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